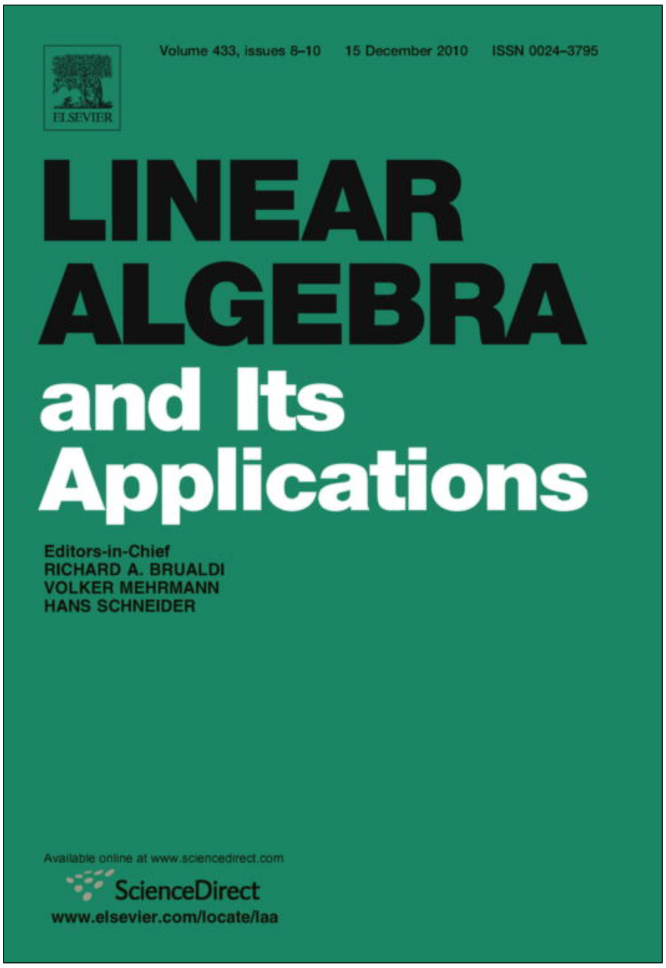
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Spectra that are Newton after extension or translation

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ABSTRACT

The appending of real numbers, and also conjugate pairs, to Newton spectra is studied to understand circumstances in which the Newton inequalities are preserved. Appending to a non-Newton spectrum to achieve the Newton inequalities is also studied. Finally the translations of Newton spectra that are Newton are also studied. A sample result is that any number of positive real numbers may be appended to a Newton spectrum, to retain the Newton property, when the Newton coefficients are positive, while any Newton spectrum may be made non-Newton by appending a conjugate pair with positive real part and sufficiently large imaginary part.

1. Introduction

A list of complex numbers $\lambda_1, \ldots, \lambda_n$ (repeats allowed), that is the spectrum of a real matrix (i.e. is self-conjugate), is called a **Newton spectrum** [2] if the normalized elementary symmetric functions

$$c_0, c_1, \ldots, c_n$$

satisfy the **Newton inequalities** [3]:

$$\Delta_k = \Delta_k(\lambda_1, \ldots, \lambda_n) = c_k^2 - c_{k-1}c_{k+1} \geqslant 0, \quad k = 1, \ldots, n-1.$$

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Here, $c_0 = 1$, and

$$\binom{n}{k} c_k = \binom{n}{k} c_k(\lambda_1, \ldots, \lambda_n) = S_k = S_k(\lambda_1, \ldots, \lambda_n) \equiv \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}, \ \ k = 1, \ldots, n.$$

The c_k's are called **Newton coefficients** and the sequence of them a **Newton sequence** if it satisfies the Newton inequalities.

We began [2] to study Newton spectra (Newton matrices) because of their connection with determinantal inequalities and potential connection with the nonnegative inverse eigenvalue problem.

As was noted by example in [2], a Newton spectrum may fail to remain Newton either when it is extended by the addition of a real number λ_{n+1} or when it is translated by a constant real number t. Our purpose here is to elaborate upon both issues. In Section 2, we consider the question of when a real number may be appended to a Newton spectrum, so that the result is Newton. We will see that this is the case when the Newton sequence is positive and the real number is positive. In fact, if the Newton sequence is positive, appending any number of positive numbers results in a Newton sequence that is also positive. In addition, in Section 3 we consider the possibility of adding a conjugate pair of complex eigenvalues to a Newton spectrum. For sufficiently large imaginary part, the result will not be Newton, but there is a trade-off with the real part. In Section 4, we consider those translations that preserve Newton sequences. This was well-developed qualitatively in [2]. After a review of prior results, we elaborate upon transitions between Newton and non-Newton spectra as a function of t. All these issues admit (different) polynomial analyses. Finally, in Section 5, we analyze which non-Newton spectra may be made Newton by appending (only) a finite number of 0's, and we determine precisely how many 0's need be appended. Other possibilities for augmentation to a Newton spectrum are also considered.

Since any sequence of real elementary symmetric functions $S_1, \ldots, S_n$ may occur, the sequence of Newton coefficients $c_1, \ldots, c_n$ may be any real sequence. In fact, a Newton sequence may have any sequence of signs, and for some sign sequences, e.g. $+ + - - + + - - \cdots$, it is easier to be Newton than for other sequences of signs, but with the same absolute values. We call a Newton sequence for which each $c_k > 0$ **positive Newton** or **p-Newton** for short. Any re-signing of a p-Newton sequence is Newton. Nonetheless, positive sequences are often easier to deal with technically, and, for the most part, prior authors have discussed only p-Newton sequences, though it has long been known [1,4] that any real spectrum is Newton (see [1] for a nice proof). Some of our results will be about positive Newton coefficients.

Thus, we recall circumstances that produce positive Newton coefficients, or, equivalently, positive $S_1, \ldots, S_n$. If $\lambda_1, \ldots, \lambda_n$ are in the right half-plane (RHP), then it is known and easy to prove inductively that

$$S_k(\lambda_1, \ldots, \lambda_n) > 0, \quad k = 1, \ldots, n$$

and, thus, that

$$c_k(\lambda_1, \ldots, \lambda_n) > 0, \quad k = 1, \ldots, n.$$

(Just use the obvious identities

$$S_k(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) = S_k(\lambda_1, \ldots, \lambda_n) + \lambda_{n+1} S_{k-1}(\lambda_1, \ldots, \lambda_n)$$

if λ_{n+1} is real and

$$\begin{aligned} S_k(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \overline{\lambda_{n+1}}\ ) &= S_k(\lambda_1, \ldots, \lambda_n) + 2Re(\lambda_{n+1}) S_{k-1}(\lambda_1, \ldots, \lambda_n) \\ &\quad + |\lambda_{n+1}|^2 S_{k-2}(\lambda_1, \ldots, \lambda_n) \end{aligned}$$

if $\lambda_{n+1}, \overline{\lambda_{n+1}}$ is a conjugate pair, with natural conventions when k is small.)

The converse, however, is not generally true, unless $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Example. The spectrum $\{3, -1 + i3, -1 - i3\}$ clearly is not in the RHP and its Newton coefficients are all positive: $c_0 = 1,\ c_1 = 1/3,\ c_2 = 4/3$ and $c_3 = 30$.

Thus, the assumption that the Newton coefficients are positive, which we often use, is more general than that the eigenvalues lie in the RHP. A matrix that has positive principal minors is a **P-matrix** and one that has positive S_k's (c_k's) is sometimes called a **Q-matrix**. It is not known whether the latter have more general spectra.

A spectrum $\lambda_1, \ldots, \lambda_n$ that does not have positive Newton coefficients may often be embedded in one that does. In fact, the spectrum may be so embedded if and only if it does not include a nonpositive real number. Just as with Newton spectra, it is difficult to describe the spectra for which the c_k's are positive. We mention several situations in which a non-Newton spectrum may be embedded in a Newton one, but in general this question seems to be open. It is likely that any one may be so embedded.

2. Real extensions of Newton spectra

Since a spectrum consisting entirely of real numbers is known to be Newton [1], extensions of a real sequence by a real number will necessarily be Newton. Moreover, it might be expected that a real extension of any Newton spectrum is Newton. Since this is not true, perhaps it will be true most of the time. We show a sense in which this is so, and give broad circumstances in which all real extensions are Newton.

Given a Newton spectrum $\lambda_1, \ldots, \lambda_{n-1}$, our analysis is to view $\Delta_k(\lambda_1, \ldots, \lambda_n)$ as a function of λ_n. As such, it turns out to be a quadratic function with coefficients involving n, k and $c_{k-2}, c_{k-1}, c_k, c_{k+1}$ all evaluated at $\lambda_1, \ldots, \lambda_{n-1}$.

Remark 1. Throughout this section we will adopt the following notation for a fixed self-conjugate spectrum $\lambda_1, \ldots, \lambda_{n-1}$:

- When c_k or Δ_k are not evaluated in a general spectrum, it will mean that they are evaluated on the fixed spectrum $\lambda_1, \ldots, \lambda_{n-1}$.
- $c_k(\lambda_n)$ and $\Delta_k(\lambda_n)$ are the kth Newton coefficient and the kth Newton difference, respectively, of the spectrum $\lambda_1, \ldots, \lambda_{n-1}, \lambda_n$, where $\lambda_n \in \mathbb{R}$.

Lemma 2. *Let $\lambda_1, \ldots, \lambda_{n-1}$ be self-conjugate and let*

$$c_k = \begin{cases} 1 & \text{for} \quad k = 0 \\ c_k(\lambda_1, \ldots, \lambda_{n-1}) & \text{for} \quad k = 1, \ldots, n-1 \\ 0 & \text{for} \quad k > n-1 \quad \text{or} \quad k < 0. \end{cases}$$

Then, for any $\lambda_n \in \mathbb{R}$

$$c_k(\lambda_1, \ldots, \lambda_n) = \frac{n-k}{n} c_k + \frac{k}{n} \lambda_n c_{k-1}, \quad k = 0, \ldots, n$$

and for $k = 1, \ldots, n-1$,

$$\begin{aligned} n^2 \Delta_k(\lambda_1, \ldots, \lambda_n) = {} & \lambda_n^2 \left[k^2 c_{k-1}^2 - (k^2 - 1) c_{k-2} c_k \right] \\ & + \lambda_n \left[(k(n-k) - (n+1)) c_k c_{k-1} - (k(n-k) - (n-1)) c_{k+1} c_{k-2} \right] \\ & + \left[(n-k)^2 c_k^2 - \left((n-k)^2 - 1 \right) c_{k+1} c_{k-1} \right]. \end{aligned}$$

In particular,

$$n^2 \Delta_1(\lambda_1, \ldots, \lambda_n) = (\lambda_n - c_1)^2 + n(n-2) \Delta_1(\lambda_1, \ldots, \lambda_{n-1}) \geqslant 0, \quad \forall \lambda_n \in \mathbb{R},$$
$$n^2 \Delta_{n-1}(\lambda_1, \ldots, \lambda_n) = (\lambda_n c_{n-2} - c_{n-1})^2 + n(n-2) \lambda_n^2 \Delta_{n-2}(\lambda_1, \ldots, \lambda_{n-1}) \geqslant 0, \quad \forall \lambda_n \in \mathbb{R}.$$

If the sequence $c_0, c_1, \ldots, c_{n-1}$ does not contain two consecutive 0's and $\lambda_1, \ldots, \lambda_{n-1}$ is a Newton spectrum, then the coefficient of λ_n^2 and the constant term for the function $n^2 \Delta_k(\lambda_1, \ldots, \lambda_n)$, when viewed as a quadratic function in λ_n, are positive.

Proof. Note that

$$S_k(\lambda_1,\ldots,\lambda_n)=\begin{cases}1 & \text{for} \quad k=0\\ S_k(\lambda_1,\ldots\lambda_{n-1})+\lambda_n S_{k-1}(\lambda_1,\ldots,\lambda_{n-1}) & \text{for} \quad k=1,\ldots,n-1\\ \lambda_n S_{n-1}(\lambda_1,\ldots,\lambda_{n-1}) & \text{for} \quad k=n.\end{cases}$$

Therefore

$$\begin{aligned} c_k(\lambda_1,\ldots,\lambda_n) &= \frac{S_k(\lambda_1,\ldots,\lambda_n)}{\binom{n}{k}}\\ &=\begin{cases}1 & \text{if} \quad k=0\\ \dfrac{\binom{n-1}{k}}{\binom{n}{k}}c_k+\dfrac{\lambda_n\binom{n-1}{k-1}}{\binom{n}{k}}c_{k-1}=\frac{n-k}{n}c_k+\frac{k}{n}\lambda_n c_{k-1} & \text{if} \quad 1\leqslant k\leqslant n-1\\ \dfrac{\lambda_n S_{n-1}(\lambda_1,\ldots,\lambda_{n-1})}{\binom{n}{n}}=\lambda_n c_{n-1} & \text{if} \quad k=n.\end{cases}\end{aligned}$$

From the definition, $\Delta_k(\lambda_1,\ldots,\lambda_n)$ is

$$\left[\frac{n-k}{n}c_k+\frac{k}{n}\lambda_n c_{k-1}\right]^2-\left[\frac{n-k+1}{n}c_{k-1}+\frac{k-1}{n}\lambda_n c_{k-2}\right]\left[\frac{n-k-1}{n}c_{k+1}+\frac{k+1}{n}\lambda_n c_k\right],$$

so that

$$\begin{aligned} n^2\Delta_k(\lambda_1,\ldots,\lambda_n) =& \lambda_n^2\left[k^2c_{k-1}^2-(k-1)(k+1)c_{k-2}c_k\right]\\ &+\lambda_n\Big[2k(n-k)c_kc_{k-1}-(k-1)(n-k-1)c_{k-2}c_{k+1}\\ &-(n-k+1)(k+1)c_{k-1}c_k\Big]\\ &+\left[(n-k)^2c_k^2-(n-k+1)(n-k-1)c_{k-1}c_{k+1}\right]\end{aligned}$$

and the expression given in the lemma is clear.

Let $k\in\{1,\ldots,n\}$. The first coefficient of $n^2\Delta_k(\lambda_1,\ldots,\lambda_n)$ as a function of λ_n is

$$k^2c_{k-1}^2-(k^2-1)c_{k-2}c_k=(k^2-1)\Delta_{k-1}(\lambda_1,\ldots,\lambda_{n-1})+c_{k-1}^2$$

which clearly is positive if $c_{k-1}\neq 0$. Otherwise, $\Delta_{k-1}(\lambda_1,\ldots,\lambda_{n-1})=-c_{k-2}c_k$ and this coefficient is also positive under the assumption about the sequence $c_0,c_1,\ldots,c_{n-1}$ not containing two consecutive 0's. A similar argument is applied to the last coefficient of $n^2\Delta_k(\lambda_1,\ldots,\lambda_n)$ as a function of λ_n. $\square$

Theorem 3. *Let $\lambda_1,\ldots,\lambda_{n-1}$, with $n\geqslant 4$, be a Newton spectrum such that the sequence $c_k=c_k(\lambda_1,\ldots,\lambda_{n-1})$, $k=0,1,\ldots,n-1$, does not contain two consecutive 0's. Then, except for λ_n lying in a collection of at most $n-3$ finite, open, real intervals, for any real λ_n, the spectrum $\lambda_1,\ldots,\lambda_n$ is also Newton. That is, $\lambda_n\notin\cup_{k=2}^{n-2}I_k$ where I_k is the open interval whose extremes are the real roots of $n^2\Delta_k(\lambda_1,\ldots,\lambda_{n-1},\lambda_n)$ when viewed as a function in λ_n if the roots are real and different, or $I_k=\emptyset$ otherwise.*

Proof. Note that the first and the last Newton inequalities for $\lambda_1,\ldots,\lambda_n$, are quadratic functions of λ_n and are always satisfied, see Lemma 2. The bound comes from the other $n-3$ Newton inequalities. $\square$

Remark 4. For $2 \leqslant k \leqslant n-2$, with the notation of the previous theorem, define

$$\begin{aligned} a_k &= k^2 c_{k-1}^2 - (k^2-1)c_{k-2}c_k \\ b_k &= (k(n-k)-(n+1))c_k c_{k-1} - (k(n-k)-(n-1))c_{k+1}c_{k-2} \\ d_k &= (n-k)^2 c_k^2 - \left((n-k)^2-1\right) c_{k+1}c_{k-1}. \end{aligned}$$

Then, when $b_k^2 > 4a_k d_k$, we have

$$I_k = \left(\frac{-b_k - \sqrt{b_k^2 - 4a_k d_k}}{2a_k}, \frac{-b_k + \sqrt{b_k^2 - 4a_k d_k}}{2a_k} \right).$$

Example. The bound given in the theorem can be attained. The spectrum $\{-1, 1 \pm i\}$ is Newton with Newton coefficients $c_0 = 1,\ c_1 = 1/3,\ c_2 = 0,\ c_3 = -2$ and $\{-1, 1 \pm i, \lambda_4\}$ is non-Newton if and only if $\lambda_4 \in (-3, -3/2)$ because $16\Delta_2(-1, 1 \pm i, \lambda_4) = \frac{2}{9}(\lambda_4+3)(2\lambda_4+3)$.

Remark 5. If $c_k = c_{k+1} = 0$ for a certain $k \in \{1, \ldots, n-2\}$, $n \geqslant 4$, then

$$\begin{aligned} n^2\Delta_{k+1}(\lambda_1, \ldots, \lambda_n) &= -\lambda_n[(k+1)(n-k-1)-(n-1)]c_{k+2}c_{k-1} \\ &= -k(n-k-2)c_{k+2}c_{k-1}\lambda_n \end{aligned}$$

is a polynomial in λ_n of degree one or zero and

$$\begin{aligned} n^2\Delta_k(\lambda_1, \ldots, \lambda_n) &= \lambda_n^2 k^2 c_{k-1}^2 \geqslant 0, \quad \forall \lambda_n \in \mathbb{R}, \\ n^2\Delta_{k+2}(\lambda_1, \ldots, \lambda_n) &= (n-k-2)^2 c_{k+2}^2 \geqslant 0, \quad \forall \lambda_n \in \mathbb{R}. \end{aligned}$$

Example. The spectrum $\left\{1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right\}$ is Newton with Newton coefficients $c_0 = c_3 = 1,\ c_1 = c_2 = 0$ and $\left\{1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \lambda_4\right\}$ is non-Newton if and only if $\lambda_4 > 0$ because $16\Delta_2\left(1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \lambda_4\right) = -\lambda_4$.

Theorem 6. *Let $\lambda_1, \ldots, \lambda_{n-1}$ be a Newton spectrum and consider $\Delta_k(\lambda_n) = \Delta_k(\lambda_1, \ldots, \lambda_n)$ as a quadratic function of λ_n, $k = 1, \ldots, n-1$. If the discriminant of $\Delta_k(\lambda_n)$ is nonpositive, $k = 2, \ldots, n-2$, then for any $\lambda_n \in \mathbb{R}$, $\lambda_1, \ldots, \lambda_n$ is a Newton spectrum.*

Lemma 7. *If $\lambda_1, \ldots, \lambda_{n-1}$ are such that*

$$c_k(\lambda_1, \ldots, \lambda_{n-1}) > 0, \quad k = 1, \ldots, n-1$$

and $\lambda_n > 0$, then

$$c_k(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) > 0, \quad k = 1, \ldots, n.$$

Proof. We have $c_k > 0$ if and only if $S_k > 0$. Since $S_k(\lambda_1, \ldots, \lambda_n) = S_k(\lambda_1, \ldots, \lambda_{n-1}) + \lambda_n S_{k-1}(\lambda_1, \ldots, \lambda_{n-1})$, $k = 1, \ldots, n$ (with $S_n(\lambda_1, \ldots, \lambda_{n-1}) = 0$), the addition of $\lambda_n > 0$ leaves the S_k's positive and, thus, the c_k's positive. □

Theorem 8. *If $\lambda_1, \ldots, \lambda_{n-1}$ is a p-Newton spectrum, then appending any number of positive real numbers will result in a p-Newton spectrum.*

Proof. It is enough to prove the result when appending one real number $\lambda_n > 0$. The expression of $n^2\Delta_k(\lambda_1, \ldots, \lambda_n)$ from Lemma 2 can be written as:

$$(k^2-1)\Delta_{k-1}\lambda_n^2+(k-1)(n-(k+1))[c_{k-1}c_k-c_{k-2}c_{k+1}]\lambda_n$$
$$+((n-k)^2-1)\Delta_k+[c_{k-1}\lambda_n-c_k]^2,$$

where $\Delta_j=\Delta_j(\lambda_1,\ldots,\lambda_{n-1})$ for $j=k-1,k$. Note that under the hypothesis of p-Newton [2, Lemma 9] we have $c_{k-1}c_k-c_{k-2}c_{k+1}\geqslant 0$ and the result is clear. $\square$

Lemma 9. *If $\lambda_1,\ldots,\lambda_{n-1}$ are such that $c_k(\lambda_1,\ldots,\lambda_{n-1})$ is a strictly alternating sequence ($c_kc_{k+1}<0, k=0,1,\ldots,n-2$) and $\lambda_n<0$ then $c_k(\lambda_1,\ldots,\lambda_{n-1},\lambda_n)$ is also a strictly alternating sequence.*

Proof. The sign of c_k is the same as the sign of S_k. Since $S_k(\lambda_1,\ldots,\lambda_n)=S_k(\lambda_1,\ldots,\lambda_{n-1})+\lambda_nS_{k-1}(\lambda_1,\ldots,\lambda_{n-1}),\ k=1,\ldots,n$ (with $S_n(\lambda_1,\ldots,\lambda_{n-1})=0$), the addition of $\lambda_n<0$ leaves the S_k's positive for k even and negative for k odd. $\square$

Because a spectrum remains Newton upon negation we have the following result:

Theorem 10. *If $\lambda_1,\ldots,\lambda_{n-1}$ is a Newton spectrum such that $c_k=c_k(\lambda_1,\ldots,\lambda_{n-1}),k=0,1,\ldots,n-1$ is a strictly alternating real sequence, then appending any number of negative real numbers results in a Newton spectrum with a strictly alternating sequence of Newton coefficients.*

Proof. It is enough to prove the result when appending one real number $\lambda_n<0$. The spectrum $-\lambda_1,\ldots,-\lambda_{n-1}$ is p-Newton, so $|c_{k-1}c_k|-|c_{k-2}c_{k+1}|\geqslant 0$ (see [2, Lemma 9]). Then, for the strictly alternating sequence c_k we have

$$c_{k-1}c_k-c_{k-2}c_{k+1}=-|c_{k-1}c_k|+|c_{k-2}c_{k+1}|\leqslant 0.$$

Since the expression of $n^2\Delta_k(\lambda_1,\ldots,\lambda_n)$ from Lemma 2 can be written as:

$$(k^2-1)\Delta_{k-1}\lambda_n^2+(k-1)(n-(k+1))[c_{k-1}c_k-c_{k-2}c_{k+1}]\lambda_n$$
$$+((n-k)^2-1)\Delta_k+[c_{k-1}\lambda_n-c_k]^2,$$

where $\Delta_j=\Delta_j(\lambda_1,\ldots,\lambda_{n-1})$ for $j=k-1,k$, the result is clear. $\square$

Since appending a real number to a Newton spectrum may cause it to fail to be Newton, we know that the union of Newton spectra need not be Newton. A more compelling example is the following:

Example. The union of Newton spectra need not be Newton. The spectra $\sigma_1=\{1,1\}$ and $\sigma_2=\{1,-1+i,-1-i\}$ ($\Delta_1(\sigma_2)=\frac{1}{9}$ and $\Delta_2(\sigma_2)=\frac{2}{3}$) are Newton but their union $\sigma_1,\sigma_2=\{1,1,1,-1+i,-1-i\}$ is not Newton because $\Delta_2(\sigma_1,\sigma_2)=\left(-\frac{1}{10}\right)^2-\frac{1}{5}\frac{1}{10}<0$.

An interesting question is to understand when the union of Newton spectra is Newton. For example, if it were always so, any spectrum could be embedded in a Newton spectrum. From Theorem 8, if one of the spectra has positive c_k's and the other consists of positive real numbers, then the union of the two (Newton) spectra is Newton. Is it the case that if both spectra are p-Newton, the union of the two spectra is p-Newton?

3. Extension of Newton spectra by a conjugate pair of complex numbers

Here, we consider appending to a self-conjugate spectrum of $n-2$ eigenvalues a conjugate pair of complex eigenvalues $x\pm iy$. Our primary purpose is to understand the circumstances under which a Newton spectrum extends to a Newton spectrum.

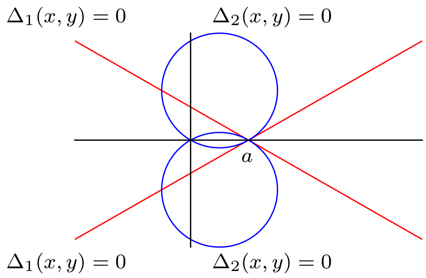

Fig. 1. Curves $\Delta_1(x,y)=\Delta_2(x,y)=0$ for $\{a\}$.

In [2], we algebraically characterized the 3-element spectra, including one complex conjugate pair, that are Newton. It is useful now to give this characterization in geometric terms. It may be thought of as appending a real eigenvalue to a conjugate pair (necessarily non-Newton) in such a way as to produce a Newton triple, but we emphasize the appending of a conjugate pair to a real number (necessarily Newton) to retain the Newton property.

Example. Consider a triple $a, x \pm iy$ with $a, x, y \in \mathbb{R}$ and $y > 0$. We consider Δ_1 and Δ_2 as functions of x and y with a fixed. Then,

$$\Delta_1(x,y) = \frac{1}{9}\left(x - a + \sqrt{3}\,y\right)\left(x - a - \sqrt{3}\,y\right)$$

and

$$\Delta_2(x,y) = \frac{1}{9}\left[\left(x-\frac{a}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}a\right)^2 - a^2\right]\left[\left(x-\frac{a}{2}\right)^2 + \left(y + \frac{\sqrt{3}}{2}a\right)^2 - a^2\right].$$

The region in which $\Delta_1(x,y) \geqslant 0$ is the region, including the x-axis, lying between the lines that pass through the point $(a,0) = (c_1(a),0)$ and form angles of $\pm\frac{\pi}{6}$ with the x-axis. The region in which $\Delta_2(x,y) \geqslant 0$ is the region either outside both or inside both of the circles with centers $\left(\frac{a}{2}, \pm\frac{\sqrt{3}\,a}{2}\right)$ and radius $|a|$. Note that these circles intersect in the points $(0,0)$ and $(a,0)$ and the lines given by $\Delta_1(x,y)=0$ are the tangents to the circles at the point $(a,0)$. See Fig. 1. Note that when $a = 0$, $\Delta_2(x,y) = \frac{1}{9}(x^2+y^2)^2 \geqslant 0$ for any pair (x,y).

To give a more general analysis, we first give formulas relating $c_k(x,y) = c_k(\lambda_1,\ldots,\lambda_{n-2}, x+iy, x-iy)$ to the $c_k(\lambda_1,\ldots,\lambda_{n-2}) = c_k$.

Remark 11. Throughout this section we will adopt the following notation for a fixed self-conjugate spectrum $\lambda_1,\ldots,\lambda_{n-2}$:

- When c_k or Δ_k or S_k are not evaluated in a general spectrum, it will mean that they are evaluated on the fixed spectrum $\lambda_1,\ldots,\lambda_{n-2}$.
- $c_k(x,y)$ and $\Delta_k(x,y)$ are the kth Newton coefficient and the kth Newton difference, respectively, of the spectrum $\lambda_1,\ldots,\lambda_{n-1}, x \pm iy$, where $x+iy \in \mathbb{C}$.

Lemma 12. *For $k = 0,1,\ldots,n$ and with the convention $c_k = 0$ if $k \notin \{0,1,\ldots,n-2\}$,*

$$n(n-1)c_k(x,y) = (n-k)(n-k-1)c_k + 2k(n-k)xc_{k-1} + k(k-1)(x^2+y^2)c_{k-2}.$$

Proof. Let $k \in \{0,1,\ldots,n\}$ and $S_k(\lambda_1,\ldots,\lambda_{n-2}) = S_k$, with the convention $S_k = 0$ if $k \notin \{0,1,\ldots,n-2\}$. Note that

$$S_k(\lambda_1,\ldots,\lambda_{n-2},x+iy,x-iy)=S_k+2xS_{k-1}+(x^2+y^2)S_{k-2}$$

and

$$\binom{n}{k}=\frac{n(n-1)}{(n-k)(n-k-1)}\binom{n-2}{k}=\frac{n(n-1)}{k(n-k)}\binom{n-2}{k-1}=\frac{n(n-1)}{k(k-1)}\binom{n-2}{k-2}$$

Therefore the result is clear. □

Also viewing $\Delta_k(\lambda_1,\ldots,\lambda_{n-2},x+iy,x-iy)$ as a function of x and y, $\Delta_k(x,y)$, we may describe $\Delta_k(x,y)$ in terms of $\lambda_1,\ldots,\lambda_{n-2}$ fixed as follows:

$$n^2(n-1)^2\Delta_k(x,y)=A_k(x^2+y^2)^2+B_kx(x^2+y^2)+C_k(x^2+y^2)+D_kx^2+E_kx+F_k$$

where

$$\begin{aligned}
A_k&=k(k-1)\left[[(k-2)(k+1)+2]c_{k-2}^2-(k-2)(k+1)c_{k-3}c_{k-1}\right]\\
B_k&=2(k-1)\left[[k^2(n-k)-k(n+1)]c_{k-2}c_{k-1}-(k-2)(k+1)(n-k-1)c_{k-3}c_k\right]\\
C_k&=k(n-k)\left[2(k-1)(n-k-1)c_{k-2}c_k-(n-k+1)(k+1)c_{k-1}^2\right]\\
&\quad-(k-1)(k-2)(n-k-1)(n-k-2)c_{k-3}c_{k+1}\\
D_k&=4\left[k^2(n-k)^2c_{k-1}^2-(k^2-1)((n-k)^2-1)c_{k-2}c_k\right]\\
E_k&=2(n-k)(n-k-1)[k(n-k)-(n+1)]c_{k-1}c_k\\
&\quad-2(k-1)(n-k+1)(n-k-1)(n-k-2)c_{k-2}c_{k+1}\\
F_k&=(n-k)(n-k-1)\left[(n-k)(n-k-1)c_k^2-(n-k+1)(n-k-2)c_{k-1}c_{k+1}\right].
\end{aligned}$$

This allows us to give a relatively simple formula for $\Delta_1(x,y)$, which is very important for our analysis.

Lemma 13. *In the notation above, we have* $(\Delta_1=\Delta_1(\lambda_1,\ldots,\lambda_{n-2}))$

$$n^2(n-1)\Delta_1(x,y)=2(n-2)(x-c_1)^2-2ny^2+n(n-2)(n-3)\Delta_1.$$

Proof. For $k=1$ we have

$$A_1=B_1=0,\quad C_1=-2n(n-1),\quad D_1=4(n-1)^2,\quad E_1=-4(n-1)(n-2)c_1$$

and

$$F_1=(n-1)(n-2)\left[(n-1)(n-2)c_1^2-n(n-3)c_0c_2\right]=(n-1)(n-2)\left[n(n-3)\Delta_1+2c_1^2\right].$$

Then,

$$\begin{aligned}
&n^2(n-1)\Delta_1(x,y)\\
&\quad=-2n(x^2+y^2)+4(n-1)x^2-4(n-2)c_1x+n(n-2)(n-3)\Delta_1+2(n-2)c_1^2\\
&\quad=2(n-2)x^2-4(n-2)c_1x-2ny^2+n(n-2)(n-3)\Delta_1+2(n-2)c_1^2\\
&\quad=2(n-2)(x-c_1)^2-2ny^2+n(n-2)(n-3)\Delta_1. \qquad \square
\end{aligned}$$

Let $\lambda_1,\ldots,\lambda_{n-2}$ be self-conjugate. Then for the region $\Delta_1(x,y)\geqslant 0$ we have:

- If $\Delta_1(\lambda_1,\ldots,\lambda_{n-2})=0$, then $\Delta_1(x,y)\geqslant 0$ is the region, including the x-axis, lying between the lines that pass through the point $(c_1,0)$ and have slopes $\pm\sqrt{\frac{n-2}{n}}$. Note that for $n=3$ the angles of these lines with the x-axis are $\pm\frac{\pi}{6}$ and as n grows these angles tend to $\pm\frac{\pi}{4}$.
- If $\Delta_1(\lambda_1,\ldots,\lambda_{n-2})=\Delta_1>0$ and $n>3$, then $\Delta_1(x,y)\geqslant 0$ is the region lying between the branches of the hyperbola $\Delta_1(x,y)=0$, that is

$$\frac{y^2}{\frac{(n-2)(n-3)}{2}\Delta_1} - \frac{(x-c_1)^2}{\frac{n(n-3)}{2}\Delta_1} = 1.$$

Note that the asymptotes of this hyperbola are the lines that pass through the point $(c_1, 0)$ and have slopes $\pm\sqrt{\frac{n-2}{n}}$. If $n = 3$, the region is the one described in the previous point.

- If $\Delta_1(\lambda_1, \ldots, \lambda_{n-2}) = \Delta_1 < 0$ and $n > 3$, then $\Delta_1(x, y) \geqslant 0$ is the region lying outside the branches of the hyperbola $\Delta_1(x, y) = 0$, that is

$$\frac{(x-c_1)^2}{\frac{n(n-3)}{2}|\Delta_1|} - \frac{y^2}{\frac{(n-2)(n-3)}{2}|\Delta_1|} = 1.$$

Again, its asymptotes are the lines that pass through the point $(c_1, 0)$ and have slopes $\pm\sqrt{\frac{n-2}{n}}$ and if $n = 3$, the region is the one described in the first point.

Now, consider not only fixed $\lambda_1, \ldots, \lambda_{n-2}$ but also fixed real part x for the appended conjugate pair $x \pm iy$. Let $\mathcal{A}(\lambda_1, \ldots, \lambda_{n-2}, x) = \mathcal{A}(x) = \{ y \in \mathbb{R} : \{\lambda_1, \ldots, \lambda_{n-2}, x + iy, x - iy\}$ is Newton$\}$. Of course, $\mathcal{A}(x)$ is symmetric about the origin. Since y^2 enters negatively into $\Delta_1(x, y)$, we have the following:

Theorem 14. *Given a Newton spectrum $\lambda_1, \ldots, \lambda_{n-2}$ and a real number x, there is a $y_0 > 0$ such that $\mathcal{A}(x) \subset [-y_0, y_0]$.*

Moreover, $\mathcal{A}(x)$ is a union of at most $2n - 3$ closed intervals and is symmetric about 0.

Proof. It is clear from the previous comments. In fact, we can take

$$y_0 \geqslant \sqrt{\frac{(n-2)(n-3)}{2}\Delta_1 + \frac{n-2}{n}(x-c_1)^2} \tag{1}$$

which is obtained intersecting $\Delta_1(x, y) = 0$ with the vertical line passing through the point $(x, 0)$.

For a fixed x, $n^2(n-1)^2\Delta_k(x, y)$ is a polynomial in y of degree at most 2 if $k = 1$ and of degree at most 4 if $k \geqslant 2$. This means that in total we have at most $2 + 4(n-2)$ real roots, therefore $\mathcal{A}(x)$ is a union of at most $1 + 2(n-2) = 2n - 3$ closed intervals. $\square$

Corollary 15. *If $\lambda_1, \ldots, \lambda_{n-2}$ is a p-Newton spectrum, then for each real number $x > 0$, the set $\mathcal{A}(x)$ is nonempty and includes an interval of positive length if all the Newton inequalities for $\lambda_1, \ldots, \lambda_{n-2}$ are strict.*

Proof. This is a particular case of Theorem 8 taking $x + iy = x - iy = x \in \mathbb{R}$. $\square$

Example. We note that the bound (1) for y_0, given in the proof of Theorem 14, is increasing in x as x moves away from $c_1 = \frac{1}{n-2}(\lambda_1 + \cdots + \lambda_{n-2})$. However, the set $\mathcal{A}(x)$ need not be an interval even when nonempty (see what follows), and when it is an interval, the length of the actual interval may not increase as we move away from c_1 (see Fig. 1 where the interval $\mathcal{A}(x)$ increases before it decreases and then increases as x moves leftward away from $a = c_1$).

The spectrum $\{4, -2\}$ is Newton and the extended spectrum $\{4, -2, x \pm iy\}$ is Newton only in the region of the complex plane outside the close curves $\Delta_2(x, y) = \Delta_3(x, y) = 0$ of Fig. 2 which is inside the region bounded by the branches of the hyperbola $\Delta_1(x, y) = 0$. Note that $\mathcal{A}(x)$ can be a closed interval or the union of three closed intervals.

The characterization of the curves $\Delta_k(x, y) = 0$, in general, is quite complex, but when the fixed spectrum has all its elements equal, $\lambda_1 = \cdots = \lambda_{n-2} = a \in \mathbb{R}$, we have for $k = n - 1$

$$n^2(n-1)\Delta_{n-1}(x, y) = 2(n-2)a^{2n-6}P(x, y)Q(x, y)$$

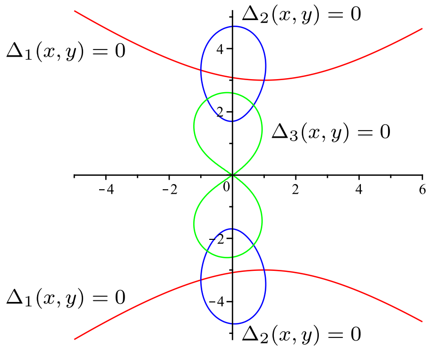

Fig. 2. Curves $\Delta_1(x,y)=\Delta_2(x,y)=\Delta_3(x,y)=0$ for $\{4,-2\}$.

where

$$P(x,y)=\left[\left(x-\frac{a}{2}\right)^2+\left(y-\sqrt{\frac{n}{4(n-2)}}\,a\right)^2-\frac{n-1}{2(n-2)}a^2\right]$$

and

$$Q(x,y)=\left[\left(x-\frac{a}{2}\right)^2+\left(y+\sqrt{\frac{n}{4(n-2)}}\,a\right)^2-\frac{n-1}{2(n-2)}a^2\right].$$

That is $\Delta_{n-1}(x,y)\geqslant 0$ is the region either outside both or inside both of the circles with centers $\left(\frac{a}{2},\pm\sqrt{\frac{n}{4(n-2)}}a\right)$ and radius $\sqrt{\frac{n-1}{2(n-2)}}|a|$.

Remark 16. Note that directly from the definitions

$$\begin{aligned}
c_n(a,\ldots,a,x+iy,x-iy)&=a^{n-2}(x^2+y^2)\\
c_{n-1}(a,\ldots,a,x+iy,x-iy)&=\frac{2a^{n-2}x+(n-2)a^{n-3}(x^2+y^2)}{n}\\
c_{n-2}(a,\ldots,a,x+iy,x-iy)&=\frac{2a^{n-2}+4(n-2)a^{n-3}x+(n-2)(n-3)a^{n-4}(x^2+y^2)}{n(n-1)}.
\end{aligned}$$

Therefore, the simplest way of calculating $\Delta_{n-1}(x,y)$ is using its definition. The expression of $\Delta_{n-1}(x,y)$, given in terms of the polynomials $P(x,y)$ and $Q(x,y)$, was obtained by observing, for small n, that $\Delta_{n-1}(x,y)=0$ can be written as the product of the equations of two circumferences, of the same radius and centers, symmetric with respect to the x-axis.

4. Translation of Newton spectra

By **translation** of a spectrum $\lambda_1,\ldots,\lambda_n$, we mean transforming it to

$$\lambda_1+t,\lambda_2+t,\ldots,\lambda_n+t$$

for some constant $t\in\mathbb{R}$, but we will view t as a real variable. Our goal is to understand more fully the translations of a Newton spectrum that are Newton. This subject was introduced in [2]. Again, since a list of real numbers is necessarily Newton, translation of any real spectrum remains Newton. However, translation of a general Newton spectrum need not remain Newton. In [2] low dimensional spectra that are forever Newton (Newton under all translations), **f-Newton**, were characterized. Also, Newton spectra with nonnegative c_k's remain Newton under all right translations, $t \geqslant 0$, (of course, the spectrum eventually becomes p-Newton), while the Newton spectra with alternating sign c_k's remain Newton under left translations ($t \leqslant 0$). Finally, the spectra, which eventually become (and stay) Newton under translation in one direction or the other, were characterized. Here, we are interested in how transitions (between Newton and non-Newton) may occur as a spectrum is translated.

Now, $\Delta_k(\lambda_1 + t, \lambda_2 + t, \ldots, \lambda_n + t) = \Delta_k(t)$ may be viewed as a polynomial of degree (at most) $2k$ in t.

Lemma 17 ([2, Lemma 12]). *The polynomial $\Delta_k(t) = \Delta_k(\lambda_1 + t, \lambda_2 + t, \ldots, \lambda_n + t)$, for $k = 1, \ldots, n-1$, is*

$$\sum_{q=0}^{2k-2} \left(\sum_{j=0}^{q} \left[\binom{k}{j}\binom{k}{q-j} - \binom{k-1}{j-1}\binom{k+1}{q+1-j} \right] c_{k-j} c_{k-(q-j)} \right) t^q.$$

where $c_j = c_j(\lambda_1, \ldots, \lambda_n)$, $j = 0, 1, \ldots, n$, and 0 otherwise.

From the formula, we see that $\Delta_k(t)$ actually has degree at most $2k-2$ and, thus, has at most $2k-2$ real roots. Each real root, depending upon the behavior of the other Δ's, might give a transition from Newton to non-Newton or vise-versa. Thus,

$$(n-1)(n-2) = \sum_{k=1}^{n-1} (2k-2)$$

is an upper bound for the number of transitions.

Theorem 18. *Suppose that $\lambda_1, \ldots, \lambda_n$ is a Newton spectrum. Then for $t \in \mathbb{R}$, $\lambda_1 + t, \ldots, \lambda_n + t$ is a Newton spectrum, except possibly for $\frac{1}{2}(n-1)(n-2)$ finite open intervals of exceptions. Equivalently, there are at most $(n-1)(n-2)$ transitions between Newton and non-Newton in t.*

We do not know a sharp bound for the number of transitions, but we do have the following evidence for $n \leqslant 4$.

Example. For $n = 2$, every Newton spectrum is real, so there are no transitions.

For $n = 3$ we can have no transitions (any real spectrum), one transition (the 3rd roots of unity with $\Delta_1(t) = 0$ and $\Delta_2(t) = -t$) and two transitions ($\{1, 4 \pm i\}$ with $\Delta_1(t) = \frac{2}{3}$ and $\Delta_2(t) = \frac{2}{3}t^2 + 8t + \frac{166}{9}$).

For $n = 4$ we can have:

- no transitions: any real spectrum or $\{1, -1, \pm i\}$ with $\Delta_1(t) = \Delta_2(t) = 0$ and $\Delta_3(t) = t^2$,
- one transition: $\{1, 3, -2 \pm i3\}$ with $\Delta_1(t) = 0$, $\Delta_2(t) = -10t$ and $\Delta_3(t) = -20t^3 - 39t^2 + 100$,
- two transitions: $\{1, -1, 1 \pm i\}$ with $\Delta_1(t) = \frac{1}{12}$, $\Delta_2(t) = \frac{1}{12}t^2 + \frac{7}{12}t + \frac{5}{18}$ and $\Delta_3(t) = \frac{1}{12}t^4 + \frac{7}{6}t^3 + \frac{31}{12}t^2 + \frac{11}{6}t + \frac{7}{12}$,
- three transitions: roots of the polynomial $x^4 - 8x^3 + 24x^2 - 8x + 1$ with $\Delta_1(t) = 0$, $\Delta_2(t) = 6t + 12$ and $\Delta_3(t) = 12t^3 + 39t^2 + 12t$ and
- four transitions: $\{4 \pm i, -4 \pm i3\}$ with $\Delta_1(t) = \frac{11}{3}$, $\Delta_2(t) = \frac{11}{3}t^2 - 16t + \frac{121}{9}$ and $\Delta_3(t) = \frac{11}{3}t^4 - 32t^3 - \frac{1154}{3}t^2 - \frac{352}{3}t + \frac{5443}{3}$.

It is not possible to have five or six transitions.

Remark 19. Any self-conjugate spectrum contained in a single vertical line, and including some complex eigenvalues, cannot be Newton.

A separate argument for the totally pure imaginary case is the following: A pure imaginary spectrum is i times a real spectrum that has as many negative as positive elements. Moreover, $c_k = i^k d_k$ if the

d_k's are the Newton coefficients for the real spectrum and the c_k's for the pure imaginary spectrum. So $c_k = 0$ for all odd k, and $c_{2k}c_{2k+2} \leqslant 0$, $k = 0, \ldots, \frac{n-2}{2}$, so that the even indexed c's strictly alternate in sign if they are nonzero. This would mean that the even indexed d's are nonnegative, which is not possible, as some of the real arguments are negative.

In the case in which the line is $Re(z) = 0$, and not all eigenvalues are real, it is a calculation to see that $\Delta_1 < 0$. Since Δ_1 is translation invariant, such a spectrum is never Newton under translation, verifying the statement of the remark.

5. Newton extensions of non-Newton spectra

In [2, Lemma 8] we proved that appending 0 to a Newton spectrum gives a Newton spectrum. Now, we ask if adding zeros to a non-Newton spectrum can make it Newton, and, if this is the case, what is the minimum number of zeros that make the spectrum Newton. That is, if $\lambda_1, \ldots, \lambda_n$ is not Newton, can we add N zeros such that $\lambda_1, \ldots, \lambda_n, \overbrace{0, \ldots, 0}^{N}$ is Newton? And what is the minimum N?

First consider small n. For $n = 1$, there is nothing to study. Note that for $n = 2$ a non-Newton spectrum is of the form $a \pm ib$ with $a, b \in \mathbb{R}$ and $b > 0$.

Theorem 20. *Let $a \pm ib$ with $a, b \in \mathbb{R}$ and $b > 0$. There is a Newton spectrum of the form $a \pm ib, 0, \ldots, 0$ if and only if $|a| > b$. In this event, the minimum number of 0's that need be appended to $a \pm ib$ to achieve Newton is*

$$\left\lceil \frac{2b^2}{a^2 - b^2} \right\rceil.$$

Proof. Let $\sigma = \{a \pm ib, \overbrace{0, \ldots, 0}^{N}\}$. We have

$$c_1(\sigma) = \frac{2a}{\binom{N+2}{1}}, \quad c_2(\sigma) = \frac{a^2 + b^2}{\binom{N+2}{2}}, \quad c_k(\sigma) = 0 \ \text{ for } \ k \geqslant 3 \ \text{ and } \ \Delta_k(\sigma) \geqslant 0 \ \text{ for } \ k \geqslant 2,$$

then, σ is Newton if and only if $\Delta_1(\sigma) \geqslant 0$. Since $b > 0$ and

$$(N + 2)^2(N + 1)\Delta_1(\sigma) = 2(a^2 - b^2)(N + 2) - 4a^2,$$

$\Delta_1(\sigma) \geqslant 0 \Longrightarrow a \neq 0$ and $a^2 - b^2 \geqslant 0$. Moreover $\Delta_1(\sigma) \geqslant 0$ and $a \neq 0 \Longrightarrow a^2 - b^2 \neq 0$. Therefore σ Newton $\Longrightarrow a^2 - b^2 > 0 \Longleftrightarrow |a| > b$.

On the other hand, if $a^2 - b^2 > 0$ then

$$\sigma \text{ is Newton} \Longleftrightarrow N \geqslant \frac{2b^2}{a^2 - b^2}$$

and

$$N = \left\lceil \frac{2b^2}{a^2 - b^2} \right\rceil$$

is the minimum number of zeros. $\square$

Theorem 21. *Let $a, b \pm ic$ be a non-Newton spectrum with $a, b, c \in \mathbb{R}$ and $c > 0$. There is a Newton spectrum of the form $a, b \pm ic, 0, \ldots, 0$ if and only if $a^2 + 2b^2 > 2c^2$ and $4(2ab + b^2 + c^2)^2 - 6a(a + 2b)(b^2 + c^2) > 0$. In this event, the minimum number of 0's that need be appended to $a, b \pm ic$ to achieve Newton is*

$$max \left\{ \left\lceil \frac{2[3c^2 - (a - b)^2]}{a^2 + 2b^2 - 2c^2} \right\rceil, \left\lceil \frac{-2[a^2(b^2 - 3c^2) - 2ab(b^2 + c^2) + (b^2 + c^2)^2]}{a^2(5b^2 - 3c^2) + 2ab(b^2 + c^2) + 2(b^2 + c^2)^2} \right\rceil \right\}.$$

Proof. Let $\sigma = \{a, b \pm ic, \overbrace{0, \ldots, 0}^{N}\}$. $a, b \pm ic$ is a non-Newton spectrum if and only if

$$(a-b)^2 < 3c^2 \quad \text{or} \quad (c^2 + b(b-a))^2 < 3a^2c^2 \qquad (*)$$

We have

$c_1(\sigma) = \frac{a+2b}{\binom{N+3}{1}}$, $c_2(\sigma) = \frac{2ab+b^2+c^2}{\binom{N+3}{2}}$, $c_3(\sigma) = \frac{a(b^2+c^2)}{\binom{N+3}{3}}$, $c_k(\sigma) = 0$ for $k \geqslant 3$ and $\Delta_k(\sigma) \geqslant 0$ for $k \geqslant 3$, then, σ is Newton if and only if $\Delta_1(\sigma) \geqslant 0$ and $\Delta_2(\sigma) \geqslant 0$. Since $c > 0$ and

$$(N+3)^2(N+2)\Delta_1(\sigma) = (a^2 + 2b^2 - 2c^2)(N+3) - (a+2b)^2,$$

$\Delta_1(\sigma) \geqslant 0 \Longrightarrow a^2 + 2b^2 - 2c^2 \geqslant 0$. Moreover if $\Delta_1(\sigma) \geqslant 0$ and $a^2 + 2b^2 - 2c^2 = 0 \Longrightarrow a + 2b = 0$ and so $(a, b) = \left(\pm\frac{2c}{\sqrt{3}}, \mp\frac{c}{\sqrt{3}}\right)$, but these two points do not satisfy condition (*), so $\Delta_1(\sigma) \geqslant 0 \Longrightarrow a^2 + 2b^2 - 2c^2 > 0$.

Since $c > 0$ and

$$(N+3)^2(N+2)^2(N+1)\Delta_2(\sigma) = (A-B)(N+2) - A, \text{ with } A = 4(2ab + b^2 + c^2)^2$$
$$\text{and } B = 6a(a+2b)(b^2+c^2),$$

$\Delta_2(\sigma) \geqslant 0 \Longrightarrow A - B \geqslant 0$. Moreover if $\Delta_2(\sigma) \geqslant 0$ and $A = B \Longrightarrow A = 0 \Longrightarrow a \neq 0$ and so $A = B = 6a(a+2b)(b^2+c^2) = 0 \Longrightarrow a + 2b = 0$. Now $a + 2b = 0$ and $A = 0 \Longrightarrow (a, b) = \left(\pm\frac{2c}{\sqrt{3}}, \mp\frac{c}{\sqrt{3}}\right)$, but these points do not verify the condition (*), so $\Delta_2(\sigma) \geqslant 0 \Longrightarrow A - B > 0$.

On the other hand, if $a^2 + 2b^2 - 2c^2 > 0$ and $A - B > 0$ then σ is Newton $\Longleftrightarrow$

$$N \geqslant \frac{(a+2b)^2}{a^2 + 2b^2 - 2c^2} - 3 \quad \text{and} \quad N \geqslant \frac{A}{A-B} - 2$$
$$\Longleftrightarrow N \geqslant max\left\{\frac{2[3c^2 - (a-b)^2]}{a^2 + 2b^2 - 2c^2}, \frac{-2[a^2(b^2 - 3c^2) - 2ab(b^2+c^2) + (b^2+c^2)^2]}{a^2(5b^2 - 3c^2) + 2ab(b^2+c^2) + 2(b^2+c^2)^2}\right\}. \quad \square$$

A similar analysis is possible, though less explicit, for general n as the next results show.

Lemma 22. *Let $\sigma^* = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ be self-conjugate and $\sigma = \{\lambda_1, \ldots, \lambda_n, \overbrace{0, \ldots, 0}^{N}\}$. Then*

$$c_k(\sigma) = \begin{cases} \frac{\binom{n}{k} c_k(\sigma^*)}{\binom{N+n}{k}} & \text{if} \quad k = 1, \ldots, n \\ 0 & \text{if} \quad k = n+1, \ldots, n+N \end{cases}$$

and for $k = 1, \ldots, N + n - 1$

$$\Delta_k(\sigma) = \frac{\binom{n}{k}^2}{\binom{N+n}{k}^2}\left[c_k(\sigma^*)^2 - \frac{(n-k)(N+n-k+1)}{(n-k+1)(N+n-k)} c_{k-1}(\sigma^*)c_{k+1}(\sigma^*)\right]$$

with the convention of $c_k(\sigma^*) = 0$ for $k \geqslant n + 1$.

Proof. Note that

$$S_k(\sigma) = \begin{cases} S_k(\sigma^*) & \text{if} \quad k = 1, \ldots, n \\ 0 & \text{if} \quad k = n+1, \ldots, n+N, \end{cases}$$

so that the expression for $c_k(\sigma)$ is clear.

Using the expression of $c_k(\sigma)$ in terms of $c_k(\sigma^*)$, we have

$$\Delta_k(\sigma) = \frac{\binom{n}{k}^2}{\binom{N+n}{k}^2} c_k(\sigma^*)^2 - \frac{\binom{n}{k-1}\binom{n}{k+1}}{\binom{N+n}{k-1}\binom{N+n}{k+1}} c_{k-1}(\sigma^*)c_{k+1}(\sigma^*).$$

Now, the identities

$$\binom{n}{k-1}\binom{n}{k+1} = \frac{k(n-k)}{(k+1)(n-k+1)}\binom{n}{k}^2$$
$$\binom{N+n}{k-1}\binom{N+n}{k+1} = \frac{k(N+n-k)}{(k+1)(N+n-k+1)}\binom{N+n}{k}^2$$

make clear the expression for the $\Delta_k(\sigma)$ in terms of the Newton coefficients for σ^*. $\square$

Note that $\Delta_k(\sigma) \geqslant 0$ for $k \geqslant n$, so in order to study the Newton character of σ it is enough to check the sign of the first $n-1$ Newton inequalities.

Theorem 23. *Let $\sigma^* = \{\lambda_1, \ldots, \lambda_n\}$ be a non-Newton spectrum. The following conditions are equivalent*:

1. *The spectrum $\sigma = \{\lambda_1, \ldots, \lambda_n, \overbrace{0, \ldots, 0}^{N}\}$ is Newton.*
2. *For $k \in \{1, \ldots, n-1\}$ with $\Delta_k(\sigma^*) < 0$ we have*
   (a) $c_k(\sigma^*) \neq 0$,
   (b) $(n-k)\Delta_k(\sigma^*) + c_k(\sigma^*)^2 > 0$ *and*
   (c) $N \geqslant -\frac{(n-k+1)(n-k)\Delta_k(\sigma^*)}{(n-k)\Delta_k(\sigma^*)+c_k(\sigma^*)^2}$.

Proof. $1 \Rightarrow 2$ Let $k \in \{1, \ldots, n-1\}$ with $\Delta_k(\sigma^*) < 0$. From Lemma 22

$$\frac{\binom{N+n}{k}^2 (n-k+1)(N+n-k)}{\binom{n}{k}^2}\Delta_k(\sigma)$$
$$= \Big[(n-k+1)(N+n-k)c_k(\sigma^*)^2 - (n-k)(N+n-k+1)c_{k-1}(\sigma^*)c_{k+1}(\sigma^*)\Big], \tag{2}$$

therefore the nonnegativity of $\Delta_k(\sigma)$ is equivalent to the nonnegativity of (2). Note that

$$(n-k+1)(N+n-k) = (n-k)(N+n-k+1) + N,$$

hence

$$(2) = (n-k)(N+n-k+1)\Delta_k(\sigma^*) + Nc_k(\sigma^*)^2 \geqslant 0. \tag{3}$$

But under the hypothesis $\Delta_k(\sigma^*) < 0$, we have $c_k(\sigma^*) \neq 0$ and

$$(3) \iff -\frac{(n-k)\Delta_k(\sigma^*) + c_k(\sigma^*)^2}{(n-k+1)(n-k)\Delta_k(\sigma^*)} \geqslant \frac{1}{N}$$

implies conditions (b) and (c).

$2 \Rightarrow 1$ Let $k \in \{1, \ldots, n-1\}$. As it was shown in the first part of the proof

$$\frac{\binom{N+n}{k}^2 (n-k+1)(N+n-k)}{\binom{n}{k}^2}\Delta_k(\sigma) = (n-k)(N+n-k+1)\Delta_k(\sigma^*) + Nc_k(\sigma^*)^2,$$

so if $\Delta_k(\sigma^*) \geqslant 0$, then $\Delta_k(\sigma) \geqslant 0$. Otherwise, $\Delta_k(\sigma^*) < 0$ and condition (c) is equivalent to the nonnegativity of $(n-k)(N+n-k+1)\Delta_k(\sigma^*) + Nc_k(\sigma^*)^2$. $\square$

In [2] was pointed out that if $\lambda_1, \ldots, \lambda_n, 0, \ldots, 0$ is a Newton spectrum with $\lambda_1, \ldots, \lambda_n$ nonzero, then $\lambda_1^{-1}, \ldots, \lambda_n^{-1}, 0, \ldots, 0$ is not necessarily Newton when adding the same number of zeros. In fact, it can happen that we can never make $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$ Newton by adding zeros.

Example. The spectrum $\sqrt{8}, 1 \pm i2$ is Newton when we add 9 zeros or more and the spectrum $\frac{1}{\sqrt{8}}, \frac{1}{5} \pm i\frac{2}{5}$ cannot be made Newton by appending zeros.

The spectrum $\sqrt{8}, 1 \pm i2$ is non-Newton because $\Delta_1 = -\frac{1}{3} - \frac{4\sqrt{2}}{9} < 0$ and $\Delta_2 = -7 - \frac{20\sqrt{2}}{9} < 0$. The application of Theorem 23 gives that the minimum number of zeros to add to make it Newton is 9:

$$c_1 = \frac{2\sqrt{2}}{3} + \frac{2}{3} \neq 0, \quad c_2 = \frac{4\sqrt{2}}{3} + \frac{5}{3} \neq 0 \quad \text{(Condition (a))}$$

$$(3-1)\Delta_1 + c_1^2 = \frac{2}{3} > 0, \quad (3-2)\Delta_2 + c_2^2 = -\frac{2}{3} + \frac{20\sqrt{2}}{9} > 0 \quad \text{(Condition (b))}$$

$$N \geqslant max\left\{\frac{-6\Delta_1}{2\Delta_1 + c_1^2}, \frac{-2\Delta_2}{\Delta_2 + c_2^2}\right\} = max\left\{3 + 4\sqrt{2}, \frac{63 + 20\sqrt{2}}{-3 + 10\sqrt{2}}\right\} = 3 + 4\sqrt{2} \approx 8.6 \quad \text{(Condition (c))}$$

The spectrum $\frac{1}{\sqrt{8}}, \frac{1}{5} \pm i\frac{2}{5}$ is non-Newton because $\Delta_1 = -\frac{7}{200} - \frac{\sqrt{2}}{90} < 0$ and the application of Theorem 23 gives that this spectrum cannot be made Newton by appending zeros:

$$(3-1)\Delta_1 + c_1^2 = -\frac{23}{600} \quad \text{(against condition (b))}.$$

Now, we ask if adding a real number or a conjugate pair of complex numbers to a non-Newton spectrum can make it Newton. That is, if $\lambda_1, \ldots, \lambda_n$ is not Newton, is there $\lambda_{n+1} \in \mathbb{R}$ or $x \pm iy \in \mathbb{C}$ such that $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}$ or $\lambda_1, \ldots, \lambda_n, x \pm iy$ is Newton?

Example. The spectrum $\sqrt{3} \pm i$ is non-Newton, but the spectra $\sqrt{3} \pm i, a$ for $a \leqslant 0$ and $\sqrt{3} \pm i, -\sqrt{3} \pm i$ are Newton because:

$$\Delta_1\left(\sqrt{3} \pm i, a\right) = \frac{a(a - 2\sqrt{3})}{9}, \qquad \Delta_2\left(\sqrt{3} \pm i, a\right) = -\frac{8\sqrt{3}(3a - 2\sqrt{3})}{27},$$

$$\Delta_1\left(\sqrt{3} \pm i, -\sqrt{3} \pm i\right) = \frac{2}{3}, \quad \Delta_2\left(\sqrt{3} \pm i, -\sqrt{3} \pm i\right) = \frac{4}{9} \quad \text{and} \quad \Delta_3\left(\sqrt{3} \pm i, -\sqrt{3} \pm i\right) = \frac{32}{3}.$$

The spectrum $1 \pm 3i,\ 1 \pm 3i, a$ is not Newton for any $a \in \mathbb{R}$:

$$\Delta_1(1 \pm 3i,\ 1 \pm 3i, a) = \frac{\left(a - 1 - 3\sqrt{5}\right)\left(a - 1 + 3\sqrt{5}\right)}{25} \geqslant 0$$

$$\Leftrightarrow a \in \left(-\infty, 1 - 3\sqrt{5}\right] \cup \left[1 + 3\sqrt{5}, +\infty\right),$$

$$\Delta_2(1 \pm 3i,\ 1 \pm 3i, a) = -\frac{8}{25}\left(a + \frac{5 + 3\sqrt{17}}{4}\right)\left(a + \frac{5 - 3\sqrt{17}}{4}\right) \geqslant 0$$

$$\Leftrightarrow a \in \left[-\frac{5+3\sqrt{17}}{4}, -\frac{5-3\sqrt{17}}{4}\right]$$
$$\text{and} \quad 1-3\sqrt{5} \approx -5.7 < -\frac{5+3\sqrt{17}}{4}$$
$$\approx -4.3 < -\frac{5-3\sqrt{17}}{4} \approx 1.8 < 1+3\sqrt{5} \approx 7.7.$$

The spectrum $1 \pm 3i, a \pm bi$ is not Newton for any $a \pm bi \in \mathbb{C}$:

$\Delta_1(1 \pm 3i, a \pm bi) = \frac{3}{2}\left(\frac{(a-1)^2}{18} - \frac{b^2}{9} - 1\right) \geqslant 0$ outside the branches of the hyperbola $\Delta_1(a,b) = 0$ and $\Delta_3(1 \pm 3i, a \pm bi) = \frac{-17(a^2+b^2)^2-20a(a^2+b^2)-100(2b^2-a^2)}{12} < 0$ in the previous region because $17(a^2+b^2)^2 + 20a(a^2+b^2) + 100(2b^2-a^2) = 17b^4 + 2b^2(17a^2+10a+100) + a^2(17a^2+20a-100)$ and $17a^2+20a-100 > 0$ for $|a-1| > \sqrt{18}$, i.e. outside the interval determined by the x-coordinates of the vertices of the hyperbola $\Delta_1(a,b) = 0$. But the spectrum $1 \pm 3i, -3, 4$ is Newton because

$$\Delta_1(1 \pm 3i, -3, 4) = \frac{9}{16}, \quad \Delta_2(1 \pm 3i, -3, 4) = \frac{21}{8} \quad \text{and} \quad \Delta_3(1 \pm 3i, -3, 4) = \frac{49}{4}.$$

Theorem 24. *For the conjugate pair $b \pm ic$, with $c > 0$, one real number a may be adjoined to give a Newton spectrum, unless $|b|/c < 1/\sqrt{3}$. In this event $\left(|b|/c < 1/\sqrt{3}\right)$, two real numbers $b \pm c$ may be adjoined to produce a Newton spectrum.*

Proof. $\sigma = \{a, b \pm ic\}$ is Newton if and only if $(a-b)^2 \geqslant 3c^2$ and $(c^2 + b(b-a))^2 \geqslant 3a^2c^2$, see [2].

On the one hand $(a-b)^2 \geqslant 3c^2 \iff a \in (-\infty, b-\sqrt{3}c] \cup [b+\sqrt{3}c, +\infty)$.

On the other hand $(c^2 + b(b-a))^2 \geqslant 3a^2c^2 \iff (b^2-3c^2)a^2 - 2b(b^2+c^2)a + (b^2+c^2)^2 \geqslant 0$, and the roots, when $b^2 - 3c^2 \neq 0$, of this quadratic expression in a are

$$\frac{b^2+c^2}{b \mp \sqrt{3}c}.$$

Note that

$$0 < b-\sqrt{3}c < \frac{b^2+c^2}{b+\sqrt{3}c} < b+\sqrt{3}c < \frac{b^2+c^2}{b-\sqrt{3}c} \quad \text{when} \quad b > \sqrt{3}c,$$
$$\frac{b^2+c^2}{b-\sqrt{3}c} < b-\sqrt{3}c < 0 < \frac{b^2+c^2}{b+\sqrt{3}c} < b+\sqrt{3}c \quad \text{when} \quad b \in \left[\frac{c}{\sqrt{3}}, \sqrt{3}c\right),$$
$$b-\sqrt{3}c < \frac{b^2+c^2}{b-\sqrt{3}c} < 0 < b+\sqrt{3}c < \frac{b^2+c^2}{b+\sqrt{3}c} \quad \text{when} \quad b \in \left(-\sqrt{3}c, -\frac{c}{\sqrt{3}}\right]$$

and

$$\frac{b^2+c^2}{b+\sqrt{3}c} < b-\sqrt{3}c < \frac{b^2+c^2}{b-\sqrt{3}c} < b+\sqrt{3}c < 0 \quad \text{when} \quad b < -\sqrt{3}c.$$

Therefore, σ is Newton in the following cases:

$b > \sqrt{3}c$ and $a \in \left(-\infty, b-\sqrt{3}c\right] \cup \left[\frac{b^2+c^2}{b-\sqrt{3}c}, +\infty\right)$;

$b = \sqrt{3}c$ and $a \in (-\infty, 0]$;

$b \in \left[\frac{c}{\sqrt{3}}, \sqrt{3}c\right)$ and $a \in \left[\frac{b^2+c^2}{b-\sqrt{3}c}, b-\sqrt{3}c\right]$;

$b \in \left(-\sqrt{3}c, -\frac{c}{\sqrt{3}}\right]$ and $a \in \left[b+\sqrt{3}c, \frac{b^2+c^2}{b+\sqrt{3}c}\right]$;

$b = -\sqrt{3}c$ and $a \in [0, +\infty)$;
$b < -\sqrt{3}c$ and $a \in \left(-\infty, \frac{b^2+c^2}{b+\sqrt{3}c}\right] \cup \left[b + \sqrt{3}c, +\infty\right)$;

and σ is not Newton if $b \in \left(-\frac{c}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ for all $a \in \mathbb{R}$; that is, if $|b|/c < 1/\sqrt{3}$.

For any $b \pm ic$ the spectrum $\sigma = \{b \pm ic, b \pm c\}$ is Newton, because $c_1(\sigma) = b, c_2(\sigma) = b^2, c_3(\sigma) = b^3, c_4(\sigma) = b^4 - c^4$ and so $\Delta_1(\sigma) = \Delta_2(\sigma) = 0$ and $\Delta_3(\sigma) = b^2c^4 \geqslant 0$. $\square$

We already know that the union of two Newton spectra need not be Newton, but what about the duplication of a Newton spectrum (the adjoining of it to itself)? That is, if $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ is a Newton (non-Newton) spectrum then is $\sigma, \sigma = \{\lambda_1, \ldots, \lambda_n, \lambda_1, \ldots, \lambda_n\}$ a Newton (non-Newton) spectrum? In the real case, it is obvious that σ is Newton if and only if σ, σ is Newton. The answer is negative for general n.

Example. The Newton inequalities for the spectrum $\sigma = \left\{\sqrt{3} \pm i, -\frac{9}{10} \pm i\right\}$ are:

$$\Delta_1 = \frac{180\sqrt{3} - 19}{1200} > 0, \quad \Delta_2 = \frac{52290\sqrt{3} - 53939}{360000} > 0 \quad \text{and}$$
$$\Delta_3 = \frac{130320\sqrt{3} - 157639}{120000} > 0.$$

Then σ is Newton, but its duplication is non-Newton because

$$\Delta_3(\sigma, \sigma) = -\frac{106517109}{490000000} + \frac{2963151\sqrt{3}}{24500000} < 0.$$

On the other, σ, σ can be a Newton spectrum with σ non-Newton. The spectrum $\sigma = \{\sqrt{3} \pm i, -\sqrt{2} \pm i\sqrt{2}\}$ is non-Newton because

$$\Delta_2(\sigma) = \frac{2\sqrt{6} - 5}{9} < 0,$$

while its duplication is Newton because all the Newton differences are positive:

$$\Delta_1(\sigma, \sigma) = \frac{2\sqrt{6} - 1}{28}, \quad \Delta_2(\sigma, \sigma) = \frac{5\sqrt{6} - 12}{49}, \quad \Delta_3(\sigma, \sigma) = \frac{16\sqrt{6} - 36}{49},$$
$$\Delta_4(\sigma, \sigma) = \frac{1472\sqrt{6} - 3248}{1225}, \quad \Delta_5(\sigma, \sigma) = \frac{256\sqrt{6} - 576}{49},$$
$$\Delta_6(\sigma, \sigma) = \frac{1280\sqrt{6} - 3072}{49}, \quad \Delta_7(\sigma, \sigma) = \frac{2048\sqrt{6} - 1024}{7}.$$

The next theorem characterize the complex case $n = 3$.

Theorem 25. *Let $\sigma = \{a, b \pm ic\}$ with $a, b, c \in \mathbb{R}$ and $c > 0$. σ is Newton if and only if its duplication is Newton.*

Proof. Without loss of generality we assume that $c = 1$, then we have

$$\Delta_1(\sigma) = \frac{(a - b)^2 - 3}{9},$$
$$\Delta_2(\sigma) = \frac{(1 + b(b - a))^2 - 3a^2}{9},$$

$$\Delta_1(\sigma,\sigma)=\frac{2((a-b)^2-3)}{45},$$
$$\Delta_2(\sigma,\sigma)=\frac{a^4+a^3b+a^2b^2-11a^2-9ab^3-13ab+6b^4-6b^2+4}{225},$$
$$\Delta_3(\sigma,\sigma)=\frac{-2a^2b^4+3b^6+4a^2+3a^4b^2-8ab^3-6ab-18a^2b^2}{225}+\frac{-6a^3b-b^2+4b^4-2a^4-2ab^5-2a^3b^3-2}{225},$$
$$\Delta_4(\sigma,\sigma)=\frac{6b^4-9a^2b^4-6a^4b^2+4b^6+a^2b^6-21a^2b^2+3ab^5+3ab^3+ab+4a^4}{225}+\frac{-9a^3b^5+ab^7+4b^2-11a^2+b^8+6a^4b^4-22a^3b^3-13a^3b+1}{225},$$
$$\Delta_5(\sigma,\sigma)=\frac{2a^2(b^2+1)^2((1+b(b-a))^2-3a^2)}{45}.$$

Since $\Delta_1(\sigma,\sigma)=\frac{2}{5}\Delta_1(\sigma)$ and $\Delta_5(\sigma,\sigma)=\frac{2}{5}a^2(b^2+1)^2\Delta_2(\sigma)$ is clear that $\Delta_1(\sigma,\sigma)\geqslant 0 \Longleftrightarrow \Delta_1(\sigma)\geqslant 0$ and $\Delta_5(\sigma,\sigma)\geqslant 0 \Longleftrightarrow \Delta_2(\sigma)\geqslant 0$ and so if the duplication of σ is Newton then σ is Newton. We need to prove that if $\Delta_i(\sigma)\geqslant 0, i=1,2$, then $\Delta_i(\sigma,\sigma)\geqslant 0$, for $i=2,3,4$. The sum of the exponents of a and b in each term in all the expressions for the Δ's are even, thus it is sufficient to prove it for $b\geqslant 0$. In this semiplane $\Delta_i(\sigma)\geqslant 0, i=1,2$, in the following cases (see Theorem 24)

$$b>\sqrt{3} \text{ and } a\in\left(-\infty,b-\sqrt{3}\right]\cup\left[\frac{b^2+1}{b-\sqrt{3}},+\infty\right);$$
$$b=\sqrt{3} \text{ and } a\in(-\infty,0];$$
$$b\in\left[\frac{1}{\sqrt{3}},\sqrt{3}\right) \text{ and } a\in\left[\frac{b^2+1}{b-\sqrt{3}},b-\sqrt{3}\right].$$

If we denote $\Delta_2(a,b)=\Delta_2(\sigma,\sigma)$, then the value of $\Delta_2(\sigma,\sigma)$ in the points of $\Delta_2(\sigma)=0$ is

$$\Delta_2\left(\frac{b^2+1}{b-\sqrt{3}},b\right)=\frac{4\left(5\sqrt{3}b^3-15b^2+5\sqrt{3}b+1\right)\left(\sqrt{3}b-1\right)^2}{225\left(b-\sqrt{3}\right)^4}.$$

This expression is nonnegative for $b\geqslant 0$ because the cubic polynomial $5\sqrt{3}b^3-15b^2+5\sqrt{3}b+1$ is strictly increasing with only one negative root $b=\frac{\sqrt[3]{5}-2}{\sqrt{3}\sqrt[3]{5}}$. Since the expression $\Delta_2(\sigma,\sigma)$ has the same (positive) sign in the positive semiplane defined by $\Delta_2(\sigma)=0$, then $\Delta_2(\sigma,\sigma)>0$. Analogously for $\Delta_3(\sigma,\sigma)$ and $\Delta_4(\sigma,\sigma)$, it is sufficient to bear in mind that

$$\Delta_3\left(\frac{b^2+1}{b-\sqrt{3}},b\right)=\frac{8(b^2+1)^2\left(5\sqrt{3}b^3-15b^2+5\sqrt{3}b-1\right)}{225\left(b-\sqrt{3}\right)^4}$$

with only one positive root $\mathbb{R}b = \frac{5-\sqrt[3]{50}}{5\sqrt{3}} < \frac{1}{\sqrt{3}}$ and

$$\Delta_4\left(\frac{b^2+1}{b-\sqrt{3}}, b\right) = \frac{4(b^2+1)^4\left(\sqrt{3}b-1\right)}{45\left(b-\sqrt{3}\right)^4}$$

so $\Delta_3(\sigma,\sigma)$ and $\Delta_4(\sigma,\sigma)$ are also positive in the positive semiplane defined by $\Delta_2(\sigma) = 0$. $\square$

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