Matrices and Spectra Satisfying the Newton Inequalities

C.R. Johnson [a], C. Marijuán [b], M. Pisonero [c, *]

[a]Dept. Mathematics, College of William and Mary, Williamsburg, Virginia, 23187 USA
[b]Dpto. Matemática Aplicada, E.T.S.I. Informática, Campus Miguel Delibes s/n, 47011-Valladolid, Spain
[c]Dpto. Matemática Aplicada, E.T.S. de Arquitectura, Avenida de Salamanca s/n, 47014-Valladolid, Spain

Abstract

An n-by-n real matrix is called a Newton matrix (and its eigenvalues a Newton spectrum) if the normalized coefficients of its characteristic polynomial satisfy the Newton inequalities.

A number of basic observations are made about Newton matrices, including closure under inversion, and then it is shown that a Newton matrix with nonnegative coefficients remains Newton under right translations. Those matrices that become (and stay) Newton under translation are characterized. In particular, Newton spectra in low dimensions are characterized.

AMS Classification: 15A15, 15A45, 11C20, 15A18.
Keywords: determinantal inequalities, Newton inequalities, Newton matrix, translation.

1 Introduction

For $A \in M_n$, we use the standard principal submatrix notation throughtout: $A[\alpha]$ means the principal submatrix of A lying in the rows and columns $\alpha \subseteq N = \{1, \ldots, n\}$. If A has eigenvalues $\lambda_1, \ldots, \lambda_n$, it is known that for $k = 1, \ldots, n$,

$$S_k(A) \equiv \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} = \sum_{|\alpha| = k} \det A[\alpha] \ ,$$

with the convention that $S_0(A) \equiv 1$ and that the characteristic polynomial of A (the polynomial whose roots are $\lambda_1, \ldots, \lambda_n$) is

$$P_A(x) = \sum_{k=0}^{n} (-1)^{n-k} S_k(A) x^{n-k} \ .$$

Of interest here are the derived quantities

$$c_k = c_k(A) = c_k(\lambda_1, \ldots, \lambda_n) = \frac{1}{\binom{n}{k}} S_k(A) \ ,$$

*Corresponding author.
E-mail addresses: crjohnso@math.wm.edu (C. R. Johnson), marijuan@mat.uva.es (C. Marijuán), mpisoner@maf.uva.es (M. Pisonero).

which we call the **Newton coefficients**. Since Newton [4, 3] observed that the **Newton inequalities**

$$c_{k-1}c_{k+1} \le c_k^2 \ , \ \ k = 1, \ldots, n-1,$$

(*i.e.* the Newton coefficients form a log-concave sequence) hold when the eigenvalues $\lambda_1, \ldots, \lambda_n$ are real, we call a matrix a **Newton matrix** (resp. the λ_k's a **Newton spectrum**) when the c_k's are real and the Newton inequalities hold. Henceforth, we assume that any spectrum $\lambda_1, \ldots, \lambda_n$ mentioned are the roots of a real polynomial (*i.e.* any complex λ_k's occur in conjugate pairs, counting multiplicities) and, without loss of generality, that any matrices are real. In general, we will say that a real sequence $c_0, c_1, \ldots, c_n$ is a **Newton sequence** if it satisfies the Newton inequalities.

Much of our analysis will be under the natural assumption that the c_k's are positive, but the Newton inequalities may hold when some of the c_k's are negative or zero.

Of course,

$$P_A(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} c_k(A) x^{n-k} \ .$$

There is, of course, a 1-1 correspondence between vectors of real Newton coefficients, real polynomials and, respectively, spectra with real elementary symmetric functions. On the other hand, there are many matrices that give the same c_k's (e.g. all similarity classes for a given spectrum). The companion matrix of the polynomial associated with the c_k's is one explicit example.

There is a long history of interest in the Newton inequalities and other modern reasons for interest, see [5]. If they hold, we know the determinantal inequalities that state that the average values of the k-by-k principal minors form a log-concave sequence. The same is true for the average k-fold product of eigenvalues. We were motivated, in part, by connections with the nonnegative inverse eigenvalue problem.

We are specially interested here in kinds of matrices that are Newton, and, correspondingly, matricial properties that Newton matrices have. Clearly, real symmetric matrices are Newton and the transpose matrix of a Newton matrix is also Newton. Because of Newton's observation, the positive semi-definite and totally nonnegative matrices are Newton (with nonnegative c_k's) and, recently, it was noted [2] that (possibly singular) $\mathcal{M}$-matrices are, as well, using the immanantal inequalities in [1]. It is possible that a matrix or spectrum be not Newton, but that a translate of it ($A + tI$ or $\lambda_1 + t, \ldots, \lambda_n + t$) be Newton. We focus upon when this happens and when all right (left) translations of a Newton matrix/spectrum remain Newton. This seems to produce a number of insights.

In the next section, we give a number of special instances of Newton spectra/matrices for future reference and then follow that with Section 3 about basic properties of Newton spectra/matrices. We then discuss which Newton sequences are preserved under translation in Section 4 and which sequences become "eventually" Newton in Section 5. One of our main results, given in that section, classifies every sequence c as either eventually Newton or never Newton.

For convenience in discussion, we will use the **Newton differences**

$$\Delta_k = c_k^2 - c_{k-1}c_{k+1} \ , \ \ k = 1, \ldots, n-1,$$

so that the Newton inequalities hold if and only if $\Delta_k \geq 0, \quad k = 1, \ldots, n-1$. Note that for a sequence of c_k's with $c_0 = 1$ we have

$$\Delta_1 = \cdots = \Delta_r = 0 \quad \Longleftrightarrow \quad c_k = \begin{cases} c_1^k & k = 1, \ldots, r+1 \quad \text{if} \quad c_1 \neq 0 \\ 0 & k = 2, \ldots, r \quad\quad \text{if} \quad c_1 = 0. \end{cases}$$

2 Special Newton Spectra and Matrices

As mentioned in the introduction,

(i) any vector of real numbers is a Newton spectrum and, thus, any matrix with real eigenvalues is a Newton matrix.

For this reason, we are particularly interested in spectra/matrices with some complex (conjugate pairs of) eigenvalues. When $n = 2$, we note that

(ii) a spectrum is Newton if and only if it is real.

Proof: For the spectrum $a \pm ib$ we have $c_0 = 1$, $c_1 = a$ and $c_2 = a^2 + b^2$. So the result is clear because $\Delta_1 = -b^2 \geq 0$ if and only if $b = 0$. ∎

Also, as noted,

(iii) any $\mathcal{M}$-matrix is Newton.

In fact, as we shall see, more is true. Any $\mathcal{M}$-matrix may be translated, at least some, to the left, retaining the Newton property.

When $n = 3$, we may also characterize the Newton spectra. If all the three eigenvalues are real the spectrum is Newton, so, we consider only the case in which there is (precisely) one conjugate pair of nonreal numbers. Suppose the three eigenvalues are:

$$a, b \pm ic$$

in which we assume, without loss of generality, $c > 0$. Then,

(iv) $a, b \pm ic$ is Newton if and only if $\begin{cases} |a - b| \geq \sqrt{3}\, c \quad \text{and} \\ |c^2 + b(b-a)| \geq \sqrt{3}\, c|a|. \end{cases}$

Proof: For this spectrum we have

$$c_0 = 1\ ,\ c_1 = \frac{a+2b}{3}\ ,\ c_2 = \frac{2ab + b^2 + c^2}{3} \quad \text{and} \quad c_3 = a(b^2 + c^2).$$

Then

$$\begin{aligned} \Delta_1 &= \frac{(a+2b)^2}{9} - \frac{2ab + b^2 + c^2}{3} = \frac{(a-b)^2 - 3c^2}{9} \\ \Delta_2 &= \frac{(2ab + b^2 + c^2)^2}{9} - \frac{a+2b}{3} a(b^2 + c^2) = \frac{(c^2 + b(b-a))^2 - 3a^2c^2}{9} \end{aligned}$$

and the result is clear because

$$\begin{aligned}\Delta_1 \geq 0 \quad &\Leftrightarrow \quad |a-b| \geq \sqrt{3}\,c \quad \text{and} \\ \Delta_2 \geq 0 \quad &\Leftrightarrow \quad |c^2 + b(b-a)| \geq \sqrt{3}\,c|a| \; .\end{aligned}$$ ∎

It is clear from the results for $n = 2$ and $n = 3$ that if a nonzero, pure imaginary conjugate pair is present, a spectrum cannot be Newton (the $n = 2$ result will be a special case of something later). That is not generally so, but for $n = 4$ it happens only in a very special situation. Let $a, b, d \in \mathbb{R}$ with $d > 0$, then

(v) the spectrum $a, b, \pm id$ is Newton if and only if $\{a, b\} = \{-d, d\}$.

Proof: It is enough to prove the result for $d = 1$. For this spectrum we have

$$c_0 = 1 \; , \; c_1 = \frac{a+b}{4} \; , \; c_2 = \frac{ab+1}{6} \; , \; c_3 = \frac{a+b}{4} \quad \text{and} \quad c_4 = ab.$$

Then

$$\begin{aligned}\Delta_1 = &\quad \left(\tfrac{a+b}{4}\right)^2 - \tfrac{ab+1}{6} \geq 0 \quad &\Leftrightarrow \; (a+b)^2 \geq \frac{8}{3}(ab+1) \\ \Delta_2 = &\quad \left(\tfrac{ab+1}{6}\right)^2 - \left(\tfrac{a+b}{4}\right)^2 \geq 0 \quad &\Leftrightarrow \; (a+b)^2 \leq \frac{4}{9}(ab+1)^2 \\ \Delta_3 = &\quad \left(\tfrac{a+b}{4}\right)^2 - \tfrac{ab+1}{6}ab \geq 0 \quad &\Leftrightarrow \; (a+b)^2 \geq \frac{8}{3}(ab+1)ab\end{aligned}$$

and the spectrum $a, b, \pm i$ is Newton if and only if

$$\max\left(\frac{8}{3}(ab+1), \; \frac{8}{3}(ab+1)ab\right) \leq (a+b)^2 \leq \frac{4}{9}(ab+1)^2. \tag{1}$$

Let us study first the case

$$\max\left(\frac{8}{3}(ab+1), \; \frac{8}{3}(ab+1)ab\right) = \frac{8}{3}(ab+1)ab.$$

This happens in one of the following situations:

- $ab + 1 = 0$. In this case,

  $$(1) \quad \Leftrightarrow \quad 0 \leq (a+b)^2 \leq 0 \quad \Leftrightarrow \quad b = -a,$$

  but we are assuming $ab + 1 = 0$, so $\{a, b\} = \{-1, 1\}$.

- $ab + 1 > 0$, so $ab \geq 1$. If (1) is verified, then

  $$\frac{8}{3}(ab+1)ab \leq \frac{4}{9}(ab+1)^2,$$

  and dividing by $\frac{8}{3}(ab+1)$ we have

  $$ab \leq \frac{1}{6}(ab+1) \quad \Rightarrow \quad ab \leq \frac{1}{5}$$

  which is a contradiction with $ab \geq 1$.

- $ab + 1 < 0$, so $ab \leq 1$. Then $ab < -1$ and a similar argument to the previous one gives a contradiction.

Let us assume now

$$\frac{8}{3}(ab+1) > \frac{8}{3}(ab+1)ab.$$

This happens in one of the following situations:

- $ab + 1 > 0$, so $ab < 1$. Then $-1 < ab < 1$ and a similar argument to the one already used gives a contradiction.

- $ab + 1 < 0$, so $ab > 1$. Then $ab < -1$ and $ab > 1$ gives a contradiction and there are no a, b in this case. ■

(vi) the spectrum $a \pm ib,\ c \pm id$, with $b > 0$ and $d > 0$, is no Newton if $a = c$ or if $ac = 0$.

Proof: For this spectrum we have

$$c_0 = 1\ ,\ c_1 = \frac{a+c}{2}\ ,\ c_2 = \frac{a^2 + 4ac + b^2 + c^2 + d^2}{6}\ ,\ c_3 = \frac{a^2c + a(c^2+d^2) + b^2c}{2} \quad \text{and}$$

$$c_4 = (a^2+b^2)(c^2+d^2).$$

Then

$$\Delta_1 = \left(\frac{a+c}{2}\right)^2 - \frac{a^2 + 4ac + b^2 + c^2 + d^2}{6} \geq 0 \iff 2(b^2+d^2) \leq (a-c)^2$$

$$\Delta_2 = \left(\frac{a^2 + 4ac + b^2 + c^2 + d^2}{6}\right)^2 - \left(\frac{a+c}{2}\right)\left(\frac{a^2c + a(c^2+d^2) + b^2c}{2}\right)$$

$$\Delta_3 = \left(\frac{a^2c + a(c^2+d^2) + b^2c}{2}\right)^2 - \frac{a^2 + 4ac + b^2 + c^2 + d^2}{6}(a^2+b^2)(c^2+d^2).$$

If $a = c$ then $\Delta_1 < 0$ because b and d are positive. If $c = 0$ then $\Delta_1 \geq 0$ and $\Delta_3 \geq 0$ are contradictory because now must be

$$a^2 \geq 2(b^2+d^2) \quad \text{and} \quad -\Delta_3 = \frac{d^2}{12}\left[2a^4 + a^2(4b^2 - d^2) + 2b^2(b^2+d^2)\right] \leq 0$$

but substituting we have

$$-\Delta_3 \geq \frac{d^2}{12}\left[2\left(2(b^2+d^2)\right)^2 + 2(b^2+d^2)(4b^2-d^2) + 2b^2(b^2+d^2)\right] = \frac{d^2(b^2+d^2)}{6}(9b^2+3d^2) > 0.$$

The case $a = 0$ is similar. ■

For a spectrum $a \pm ib,\ c \pm id$, with $b, d > 0,\ a \neq c$ and $ac \neq 0$ we have

$$\Delta_1 = \left(\frac{a+c}{2}\right)^2 - \frac{a^2 + 4ac + b^2 + c^2 + d^2}{6} \geq 0 \iff 2(b^2 + d^2) \leq (a-c)^2.$$

This inequality can be seen as the points (b, d) in the disc with center $(0, 0)$ and radius $|a - c|/\sqrt{2}$. The other two Newton inequalities, $\Delta_2, \Delta_3 \geq 0$, are compatible in part of this circle but their characterization requires expressions quite complicate.

It follows from (i) that if a spectrum is real, not only is it Newton, but any translate of it is as well. This characterizes real spectra for $n < 4$ as seen from (ii) and (iv), but we will see that this is not so for $n = 4$, though there is only a very special exception coming from (v).

We close this section by considering the case in which $A = tI + C_n$ with C_n the basic circulant matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

whose eigenvalues are the n-th roots of unity. This builds upon (v) with $d = 1$. Matricially, we have

$$\begin{aligned} c_0 &= 1 \\ c_1 &= t \\ c_2 &= t^2 \\ &\vdots \\ c_{n-1} &= t^{n-1} \\ c_n &= t^n + (-1)^{n+1}. \end{aligned}$$

Thus, the first $n-2$ Newton differences are $\Delta_1 = \cdots = \Delta_{n-2} = 0$ and the last is $(-1)^n t^{n-2}$. Thus,

(vii) the Newton inequalities hold for $tI + C_n$ for any t when n is even and for all $t \leq 0$ when n is odd.

3 Basic General Ideas

We record here a number of basic facts about the Newton inequalities and sequence for which they hold. Perhaps the most basic (follows from the quadratic homogeneity of the inequalities) is

Lemma 1. *If the Newton inequalities hold for the Newton coefficients* $c_0, c_1, \ldots, c_n$ *and* h *is a real constant, the Newton inequalities also hold for* $hc_0, hc_1, \ldots, hc_n$.

Lemma 2. *The following two sequences satisfy the Newton inequalities:*

(a) *Any sequence satisfying:* $|c_0| = |c_1| = \cdots = |c_n|$.

(b) $c_0 = 1, c_1 = b, c_2 = b^2, \ldots, c_n = b^n$ *for any* $b \in \mathbb{R}$.

Examples. Sequences with the sign patterns $++--++--\cdots, +--++--\cdots$, etc are Newton. If $c_0, \ldots, c_n$ is Newton so is $c_0, \ldots, c_n, 0$.

Lemma 3. *Let* $c_0, \ldots, c_n$ *and* $d_0, \ldots, d_n$ *be sequences satisfying the Newton inequalities. Suppose that* $d_k \geq 0$, $k = 0, \ldots, n$ *and the sequence* $e_0, \ldots, e_n$ *is defined by* $c_k = e_k|c_k|$, $k = 0, \ldots, n$, *with* $e_k = \pm 1$ *if* $c_k = 0$. *Then, for any positive p and q, the sequence*

$$e_0\,|c_0|^p\,d_0^q,\ e_1\,|c_1|^p\,d_1^q,\ \ldots,\ e_n\,|c_n|^p\,d_n^q$$

is Newton.

It follows from Lemmas 2 (a) and 3 that

Corollary 4. *If* $e_0, e_1, \ldots, e_n$ *is a sequence of* ± 1*'s and* $c_0, c_1, \ldots, c_n \geq 0$ *is any sequence satisfying the Newton inequalities, then*

$$e_0\,c_0,\ e_1\,c_1,\ \ldots,\ e_n\,c_n$$

satisfies the Newton inequalities.

We note that in another case, that in which the c_k's alternate, $+-+-\cdots$, the c_k's may be replaced by their absolute values to preserve Newton. In general, the nonnegative assumption above is necessary.

Corollary 5. *If* $\lambda_1, \ldots, \lambda_n$ *is a Newton spectrum (resp. A is a Newton matrix) and h is a real constant, then* $h\lambda_1, \ldots, h\lambda_n$ *is a Newton spectrum (resp. hA is a Newton matrix).*

Proof: Lemma 2 (b) and Lemma 3. ∎

Lemma 6. *If the sequence* $c_0, c_1, \ldots, c_n$ *satisfies the Newton inequalities, then so does the sequence* $c_n, c_{n-1}, \ldots, c_1, c_0$. *If A is a Newton matrix, so is any similarity of A. If* $\lambda_1, \ldots, \lambda_n$ *is a Newton spectrum, so is any permutation of* $\lambda_1, \ldots, \lambda_n$.

Theorem 7. *If A is an invertible Newton matrix (resp.* $\lambda_1, \ldots, \lambda_n$ *is a totally nonzero Newton spectrum), then* A^{-1} *is Newton (resp.* $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$ *is a Newton spectrum).*

Proof: Note that

$$S_k(\lambda_1^{-1}, \ldots, \lambda_n^{-1}) = \frac{1}{\det A} S_{n-k}(\lambda_1, \ldots, \lambda_n), \quad \text{for} \quad k = 0, \ldots, n.$$

Then

$$c_k(A^{-1}) = \frac{1}{\det A} c_{n-k}(A), \quad \text{for} \quad k = 0, \ldots, n$$

and the result follows from Lemmas 1 and 6. ∎

In [2] it was shown that both $\mathcal{M}$-matrices and *inv*$\mathcal{M}$-matrices are Newton. We note that either of these statements actually follows inmediatelly from the other, by Theorem 7.

Lemma 8. *If* $\lambda_1, \ldots, \lambda_n$ *is a Newton spectrum, then so is* $\lambda_1, \ldots, \lambda_n, 0$ *.*

Proof: Note that

$$S_k(\lambda_1, \ldots, \lambda_n, 0) = \begin{cases} S_k(\lambda_1, \ldots, \lambda_n) & \text{if } \quad k \leq n, \\ 0 & \text{if } \quad k = n+1. \end{cases}$$

Let σ be the spectrum $\lambda_1, \ldots, \lambda_n, 0$ and σ^* the spectrum $\lambda_1, \ldots, \lambda_n$. Therefore

$$c_k(\sigma) = \frac{1}{\binom{n+1}{k}} S_k(\sigma) = \begin{cases} \dfrac{n+1-k}{(n+1)\dbinom{n}{k}} S_k(\sigma^*) = \dfrac{n+1-k}{n+1} c_k(\sigma^*) & \text{if } \quad k \leq n, \\ 0 & \text{if } \quad k = n+1. \end{cases}$$

For $k \leq n-1$ we have

$$c_k(\sigma)^2 - c_{k-1}(\sigma)c_{k+1}(\sigma) = \left(\frac{n+1-k}{n+1}\right)^2 c_k(\sigma^*)^2 - \frac{n+1-k+1}{n+1} c_{k-1}(\sigma^*) \frac{n+1-k-1}{n+1} c_{k+1}(\sigma^*) =$$

$$\left(\frac{n+1-k}{n+1}\right)^2 c_k(\sigma^*)^2 - \frac{(n+1-k)^2-1}{(n+1)^2} c_{k-1}(\sigma^*)c_{k+1}(\sigma^*) =$$

$$\frac{(n+1-k)^2-1}{(n+1)^2}\Big(c_k(\sigma^*)^2 - c_{k-1}(\sigma^*)c_{k+1}(\sigma^*)\Big) + \frac{1}{(n+1)^2} c_k(\sigma^*)^2 \geq 0.$$

Finally, for $k = n$, we have $c_n(\sigma)^2 - c_{n-1}(\sigma)c_{n+1}(\sigma) = c_n(\sigma)^2 \geq 0$ and the result is proved. ■

Example. There is no converse to Lemma 8. The spectrum $b \pm ic, 0$ with $c > 0$ is Newton, see condition (*iv*), if and only if $|b| \geq c\sqrt{3}$ and the spectrum $b \pm ic$ is never Newton, see condition (*ii*).

If $\lambda_1, \ldots, \lambda_r, 0, \ldots, 0$ is a Newton spectrum with $\lambda_1, \ldots, \lambda_r$ nonzero, what about the spectrum $\lambda_1^{-1}, \ldots, \lambda_r^{-1}, 0, \ldots, 0$? Is there a reversal principal to prove this? No.

Example. The spectrum $-5, \sqrt{2}+i\sqrt{2}, \sqrt{2}-i\sqrt{2}, 0$ is Newton:

$$c \,:\, 1,\ \frac{2\sqrt{2}-5}{4},\ \frac{2-5\sqrt{2}}{3},\ -5,\ 0 \quad \Rightarrow \quad \Delta_1 = \frac{67+20\sqrt{2}}{48},\ \Delta_2 = \frac{10\sqrt{2}-9}{36},\ \Delta_3 = 25,$$

while the spectrum $-\frac{1}{5}, \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4} + i\frac{\sqrt{2}}{4}, 0$ is not Newton:

$$c \,:\, 1,\ \frac{5\sqrt{2}-2}{40},\ \frac{5-2\sqrt{2}}{120},\ -\frac{1}{80},\ 0 \quad \Rightarrow \quad \Delta_1 = \frac{10\sqrt{2}-19}{2400} < 0.$$

If $c_0, c_1, \ldots, c_n$ are nonnegative Newton coefficients satisfying the Newton inequalities, then certain extended Newton inequalities also hold.

Lemma 9. *Let* $c_0, c_1, \ldots, c_n \geq 0$ *satisfy the Newton inequalities. Then, we have*

$$c_r\, c_s \leq c_p\, c_q$$

whenever $0 \leq r \leq p \leq q \leq s \leq n$ *and* $r+s = p+q$.

Proof: If $p=0$ or $q=n$, we have the equality $c_r\, c_s = c_p\, c_q$. In other case, we consider the sequence of Newton inequalities

$$c_{k-1}c_{k+1} \leq c_k^2, \quad \text{for} \quad 1 \leq p \leq k \leq q \leq n-1.$$

Because the c_k's are nonnegative, multiplying these inequalities we have

$$\prod_{k=p}^{q} c_{k-1}c_{k+1} = c_{p-1}c_p \left(\prod_{k=p+1}^{q-1} c_k^2 \right) c_q c_{q+1} \leq \prod_{k=p}^{q} c_k^2\,,$$

and simplifying we obtain

$$c_{p-1}c_{q+1} \leq c_p c_q, \quad \text{for} \quad 1 \leq p \leq q \leq n-1.$$

Using recursively this inequality we get the result:

$$c_r c_s = c_{p-(p-r)}c_{q+(p-r)} \leq c_p c_q. \qquad \blacksquare$$

We note that the Newton inequalities are a special case.

Examples. If $\lambda_1, \ldots, \lambda_{n-1}$ is Newton, can $\lambda_1, \ldots, \lambda_n$ not be? Yes, both $-1, 1 \pm i$ and $2, -1 \pm i\sqrt{3}$ are Newton and neither of $-1, 1 \pm i, -2$ or $2, -1 \pm i\sqrt{3}, 1$ are Newton.

Given any $\lambda_1, \ldots, \lambda_{n-1}$, the spectrum of a real matrix, is there a real number λ_n such that $\lambda_1, \ldots, \lambda_n$ is a Newton spectrum? Not always, as is shown by $\pm i$. Note that $\pm i, \lambda$ is never Newton for λ real because of condition (iv) in Section 2.

Examples. It can happen that all $(n-1)$-by-$(n-1)$ principal submatrices are Newton but A is not. This is the situation for

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

It can also happen that no $n-1$ principal submatrices are Newton, but A is. This is the case for the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

4 Translatability of Newton Sequences

Given the Newton coefficients $c : c_0, \ldots, c_n$ determined by $\lambda_1, \ldots, \lambda_n$, let $c(t) : c_0(t), \ldots, c_n(t)$ be the Newton coefficients determined by $\lambda_1 + t, \ldots, \lambda_n + t$. We refer to $c(t)$ as a left (right) translation if $t < 0$ (if $t > 0$). If c is a Newton sequence $c(t)$ may, or may not, also be a Newton sequence. The Newton differences for $c(t)$ will be denotated by $\Delta_k(t)$, *i.e.*

$$\Delta_k(t) = c_k(t)^2 - c_{k-1}(t)c_{k+1}(t) \quad , \quad k = 1, \ldots, n-1.$$

Example. Consider

$$A(t) = \begin{pmatrix} t & 1 & 0 \\ 0 & t & 1 \\ 1 & 0 & t \end{pmatrix} \qquad \begin{array}{l} c_0(t) = 1 \\ c_1(t) = t \\ c_2(t) = t^2 \\ c_3(t) = t^3 + 1 \end{array} \qquad \begin{array}{l} \Delta_1(t) = 0 \\ \Delta_2(t) = -t. \end{array}$$

In particular, the 3rd roots of the unity are Newton but a right translation is not Newton and a left translation is Newton.

We call a Newton sequence c **right** (resp. **left**) **translatable** if $c(t)$ is a Newton sequence for all $t > 0$ (all $t < 0$). We further call a Newton sequence c **forever Newton** (f-Newton) if $c(t)$ is Newton for all t. Of course a sequence coming from real λ's is f-Newton.

Theorem 10. *If $n < 4$, then a sequence c is f-Newton if and only if c is a Newton sequence resulting from real λ's.*

Proof: The result is clear for $n = 1$. For $n = 2$ is a consequence of condition (*ii*) in Section 2. For $n = 3$, the only Newton spectra are the real ones or the ones satisfying condition (*iv*) in Section 2, *i.e.* for $c > 0$

$$a, b \pm ic \quad \text{is Newton} \quad \iff \quad \begin{cases} |a-b| \geq \sqrt{3}\, c \quad \text{and} \\ |c^2 + b(b-a)| \geq \sqrt{3}\, c|a|. \end{cases}$$

Note that the first condition is the same for the spectrum $a, b \pm ic$ as for the spectrum $a+t, b+t \pm ic$, with t real. Let us see that the second condition is not true for $t = -b$

$$|c^2 + (b+t)(b+t-(a+t))| = c^2 < 3c^2 = \sqrt{3}\, c\sqrt{3}\, c \leq \sqrt{3}\, c|a-b| = \sqrt{3}\, c|a+t|$$

where the last inequality is due to the first condition. So the theorem is proved for $n = 3$. ∎

Example. Note that for $n = 4$, the basic circulant matrix shows that non-real spectrum may be f-Newton.

Theorem 11. *For $n = 4$ a sequence c is f-Newton if and only if the λ's are real or a translation of the λ's is of the form $h(-1, -i, i, 1)$ with real h.*

Proof: $\Rightarrow$) Let us assume that the f-Newton sequence is not resulting from real λ's. Then, the spectrum can have two real numbers or none.

If the spectrum $a \pm ib, c \pm id$ with $b, d > 0$ is Newton, then $a \neq 0$ by condition (vi) in Section 2. This spectrum can not be f-Newton because the translated spectrum $\pm ib, c - a \pm id$ is not Newton by condition (vi) in Section 2.

If the spectrum $a, b, c \pm id$ is f-Newton then the translated spectrum $a - c, b - c, \pm id$ is Newton and by condition (v) in Section 2 has to be of the form $h(-1, -i, i, 1)$.

$\Leftarrow$) If the λ's are real the result is clear by condition (ii) in Section 2. Let us see that any translation of a spectrum of the form $h(-1, -i, i, 1)$ is Newton. In this case we have $c(t) : 1, t, t^2, t^3, t^4 - h^4$, therefore $\Delta_1(t) = \Delta_2(t) = 0$ and $\Delta_3(t) = h^4 t^2 \geq 0$. ■

We note that $c_k(t)$ is a polynomial in t (of degree k)

$$c_k(t) = \sum_{j=0}^{k} \binom{k}{j} c_{k-j} t^j \ , \quad k = 0, \ldots n,$$

which means that $\Delta_k(t)$ is also a polynomial in t, of degree at most $2k$. The degree may be less than $2k$ if there is cancellation in the leading term of the two polynomials $c_k(t)^2$ and $-c_{k-1}(t)c_{k+1}(t)$. It is easy to see that the lead term of each polynomial is the same, so that there will always be some cancellation. There may be more, but generically this is all.

In what follows, we understand that the combinatorial number

$$\binom{m}{j} = 0$$

if m and j are integers with $m \geq 0$ and $j < 0$ or $j > m$.

Lemma 12. *The polynomial $\Delta_k(t)$, for $k = 1, \ldots, n-1$, is*

$$\sum_{q=0}^{2k-2} \left(\sum_{j=q-k}^{k} \left[\binom{k}{j}\binom{k}{q-j} - \binom{k-1}{j-1}\binom{k+1}{q+1-j} \right] c_{k-j} c_{k-(q-j)} \right) t^q.$$

In particular, it has degree at most $2k-2$ and the coefficient of t^{2k-2} is $c_1^2 - c_0 c_2$.

Proof: We have

$$\Delta_k(t) = \left[\sum_{j=0}^{k} \binom{k}{j} c_{k-j} t^j \right]^2 - \left[\sum_{j=0}^{k-1} \binom{k-1}{j} c_{k-1-j} t^j \right] \left[\sum_{j=0}^{k+1} \binom{k+1}{j} c_{k+1-j} t^j \right] =$$

$$\sum_{q=0}^{2k} \left[\sum_{j=0}^{q} \binom{k}{j}\binom{k}{q-j} c_{k-j} c_{k-(q-j)} - \sum_{j=0}^{q} \binom{k-1}{j}\binom{k+1}{q-j} c_{k-1-j} c_{k+1-(q-j)} \right] t^q =$$

$$\sum_{q=0}^{2k} \left[\sum_{j=q-k}^{k} \binom{k}{j}\binom{k}{q-j} c_{k-j} c_{k-(q-j)} - \sum_{j=q-k-1}^{k-1} \binom{k-1}{j}\binom{k+1}{q-j} c_{k-1-j} c_{k+1-(q-j)} \right] t^q =$$

$$\sum_{q=0}^{2k} \left(\sum_{j=q-k}^{k} \left[\binom{k}{j}\binom{k}{q-j} - \binom{k-1}{j-1}\binom{k+1}{q+1-j} \right] c_{k-j} c_{k-(q-j)} \right) t^q.$$

Then the coefficient of t^{2k} is

$$\left[\binom{k}{k}\binom{k}{k} - \binom{k-1}{k-1}\binom{k+1}{k+1}\right] c_0c_0 = 0$$

and the coefficient of t^{2k-1} is

$$\left[\binom{k}{k-1}\binom{k}{k} - \binom{k-1}{k-2}\binom{k+1}{k+1}\right] c_1c_0 + \left[\binom{k}{k}\binom{k}{k-1} - \binom{k-1}{k-1}\binom{k+1}{k}\right] c_0c_1 = 0.$$

The coefficient of t^{2k-2} is

$$\sum_{j=k-2}^{k} \left[\binom{k}{j}\binom{k}{2k-2-j} - \binom{k-1}{j-1}\binom{k+1}{2k-1-j}\right] c_{k-j}c_{j-k+2}.$$

Grouping together the terms c_0c_2 and c_2c_0 and using the identity $\binom{m}{m-j} = \binom{m}{j}$ we have

$$\left[\binom{k}{1}^2 - \binom{k-1}{1}\binom{k+1}{1}\right] c_1^2 + \left[2\binom{k}{0}\binom{k}{2} - \binom{k-1}{2}\binom{k+1}{0} - \binom{k-1}{0}\binom{k+1}{2}\right] c_0c_2.$$

Finally, we obtain that the coefficient of t^{2k-2} is $c_1^2 - c_0c_2$. ■

Note that the central term of $\Delta_k(t)$ is the one of degree $k-1$ and there is a certain symmetry between the coefficients of t^{k-1+p} and t^{k-1-p} for $p = 1, \ldots, k-1$. The polynomial $\Delta_k(t)$ has the form

$$(c_1^2 - c_0c_2)t^{2k-2} + (k-1)(c_1c_2 - c_0c_3)t^{2k-3} + (k-1)\left[\frac{k}{2}c_2^2 - c_1c_3 - \frac{k-2}{2}c_0c_4\right] t^{2k-4} + \cdots +$$

$$+\left(\sum_{j=1}^{k}\left[\binom{k}{j}\binom{k}{k-1-j} - \binom{k-1}{j-1}\binom{k+1}{k-j}\right] c_{k-j}c_{j+1}\right) t^{k-1} +$$

$$+ \cdots + (k-1)\left[\frac{k}{2}c_{k-1}^2 - c_{k-2}c_k - \frac{k-2}{2}c_{k-3}c_{k+1}\right] t^2 + (k-1)(c_{k-1}c_k - c_{k-2}c_{k+1})t + (c_k^2 - c_{k-1}c_{k+1}).$$

We will use the above lemma here, but also again in the next section. We first note that if c is a Newton sequence, it may happen that some right translations are not Newton (the basic circulant matrix 3-by-3 Newton sequence) or that some left translations are not Newton (the spectrum $1, 2 \pm i\frac{\sqrt{3}}{3}$ is Newton but no left translation, with $t < -\frac{5}{3}$, is because $\Delta_2(t) = \frac{8}{27}t + \frac{40}{81}$). We know of no example in which both some (but not all) left and some (but not all) right translations fail to be Newton. More precisely, what partitions of the real line into points t, for which $c(t)$ is Newton and points t for which $c(t)$ is not Newton, are possible? We may prove the following, one of our main results.

Theorem 13. *If c is a nonnegative Newton sequence, then $c(t)$ is a Newton sequence for all $t \geq 0$.*

Proof: Let us prove, in three parts, that all the coefficients of the polynomial $\Delta_k(t) = c_k(t)^2 - c_{k-1}(t)c_{k+1}(t)$ are nonnegative.

Part one. The coefficient of t^{k+p}, for $0 \le p \le k-2$, is nonnegative.

We can rewrite the coefficient of t^q given in Lemma 12, for $k \le q \le 2k-2$, replacing q by $k+p$

$$\sum_{i=0}^{k-p} \left[\binom{k}{p+i}\binom{k}{k-i} - \binom{k-1}{p+i-1}\binom{k+1}{k-i+1} \right] c_{k-p-i}c_i, \quad \text{for} \quad 0 \le p \le k-2.$$

The proof depends on the parity of $k+p$.

a) If $k+p$ is even, we now rewrite the coefficient of t^{k+p} grouping together the terms $c_{k-p-i}c_i$ and c_ic_{k-p-i} in the following way

$$\left[\binom{k}{\frac{k+p}{2}}\binom{k}{\frac{k+p}{2}} - \binom{k-1}{\frac{k+p}{2}-1}\binom{k+1}{\frac{k+p}{2}+1} \right] c_{\frac{k-p}{2}} c_{\frac{k-p}{2}} + \sum_{m=1}^{\frac{k-p}{2}} r_m c_{\frac{k-p}{2}-m} c_{\frac{k-p}{2}+m}$$

where

$$r_m = 2\binom{k}{\frac{k+p}{2}-m}\binom{k}{\frac{k+p}{2}+m} - \binom{k-1}{\frac{k+p}{2}-m-1}\binom{k+1}{\frac{k+p}{2}+m+1} - \binom{k-1}{\frac{k+p}{2}+m-1}\binom{k+1}{\frac{k+p}{2}-m+1}. \tag{2}$$

We now denote by R_m, for $m = 0, 1, \ldots, \frac{k-p}{2}$, the partial sums of the combinatorial coefficients r_m of $c_{\frac{k-p}{2}} c_{\frac{k-p}{2}}, c_{\frac{k-p}{2}-1} c_{\frac{k-p}{2}+1}, \ldots, c_0 c_k$, *i.e.*

$$R_0 = r_0 = \binom{k}{\frac{k+p}{2}}\binom{k}{\frac{k+p}{2}} - \binom{k-1}{\frac{k+p}{2}-1}\binom{k+1}{\frac{k+p}{2}+1}, \quad R_m = \sum_{i=1}^{m} r_i = R_{m-1} + r_m, \quad 1 \le m \le \frac{k-p}{2},$$

$R_{\frac{k-p}{2}}$ being the sum of all combinatorial coefficients of t^{k+p}. With this notation, we can write the coefficient of t^{k+p} as

$$R_0 c_{\frac{k-p}{2}} c_{\frac{k-p}{2}} + \sum_{m=1}^{\frac{k-p}{2}} (R_m - R_{m-1}) c_{\frac{k-p}{2}-m} c_{\frac{k-p}{2}+m}.$$

The value of $r_m = R_m - R_{m-1}$ can be negative, but we know by Lemma 9 that

$$c_{\frac{k-p}{2}-r} c_{\frac{k-p}{2}+r} \ge c_{\frac{k-p}{2}-s} c_{\frac{k-p}{2}+s}, \quad \text{for all} \quad 0 \le r < s,$$

and so, for the coefficient of t^{k+p} to be nonnegative, it is sufficient that the partial sums below are nonnegative:

$$R_0, \quad \sum_{i=0}^{m} (R_i - R_{i-1}) = R_m, \quad \text{for} \quad 1 \le m \le \frac{k-p}{2}.$$

Then, let us see that the sequence R_m, for $0 \leq m \leq \frac{k-p}{2}$, is nonnegative. More exactly, we will see that the sequence R_m is positive for $0 \leq m \leq \frac{k-p}{2} - 1$ and zero for $m = \frac{k-p}{2}$.

The additive law of recurrence for the sequence R_m given above can also be written as a multiplicative law of recurrence in the following way (see part three)

$$R_0 = r_0 = \binom{k}{\frac{k+p}{2}}\binom{k}{\frac{k+p}{2}} - \binom{k-1}{\frac{k+p}{2}-1}\binom{k+1}{\frac{k+p}{2}+1} = \binom{k}{\frac{k+p}{2}}^2 \frac{k-p}{k(k+p+2)}, \tag{3}$$

$$R_m = R_{m-1} \cdot \frac{(2m+1)(k+p-2m+2)(k-p-2m)}{(2m-1)(k+p+2m+2)(k-p+2m)}, \quad \text{for} \quad 1 \leq m \leq \frac{k-p}{2}. \tag{4}$$

and using this multiplicative law of recurrence, we can write the general term of the sequence R_m in the form

$$R_m = \binom{k}{\frac{k+p}{2}}^2 \frac{(2m+1)(k-p)}{k(k+p+2)} \prod_{j=1}^{m} \frac{(k+p+2-2j)(k-p-2j)}{(k+p+2+2j)(k-p+2j)}. \tag{5}$$

It is obvious that both R_0 and R_m, for $1 \leq m \leq \frac{k-p}{2} - 1$, are positive and that $R_{\frac{k-p}{2}}$ is zero. Now the result follows from Lemma 9.

b) If $k+p$ is odd, we write the coefficient of t^{k+p} grouping together the terms $c_{k-p-i}c_i$ and c_ic_{k-p-i} as follows

$$\sum_{m=0}^{\frac{k-p-1}{2}} r_m c_{\frac{k-p-1}{2}-m} c_{\frac{k-p+1}{2}+m}$$

where

$$r_m = 2\binom{k}{\frac{k+p-1}{2}-m}\binom{k}{\frac{k+p+1}{2}+m} - \binom{k-1}{\frac{k+p-1}{2}-m-1}\binom{k+1}{\frac{k+p+1}{2}+m+1} - \binom{k-1}{\frac{k+p+1}{2}+m-1}\binom{k+1}{\frac{k+p-1}{2}-m+1}.$$

The construction of the sequence R_m, for $m = 0, 1, \ldots, \frac{k-p-1}{2}$, of the partial sums of the combinatorial coefficients of $c_{\frac{k-p-1}{2}} c_{\frac{k-p+1}{2}}, c_{\frac{k-p-1}{2}-1} c_{\frac{k-p+1}{2}+1}, \ldots, c_0 c_k$ and the process to obtain the general term of this sequence are similar to the even case. In the odd case, the multiplicative law of recurrence for the sequence R_m is

$$\begin{aligned} R_0 = 2&\binom{k}{\frac{k+p-1}{2}}\binom{k}{\frac{k+p+1}{2}} - \binom{k-1}{\frac{k+p-1}{2}-1}\binom{k+1}{\frac{k+p+1}{2}+1} - \binom{k-1}{\frac{k+p+1}{2}-1}\binom{k+1}{\frac{k+p-1}{2}+1} \\ &= \binom{k}{\frac{k+p-1}{2}}^2 \frac{2(k-p+1)(k-p-1)}{k(k+p+1)k+p+3)}, \end{aligned} \tag{6}$$

$$R_m = R_{m-1} \cdot \frac{(m+1)(k+p-2m+1)(k-p-2m-1)}{m(k-p+2m+1)(k+p+2m+3)}, \quad \text{for} \quad 1 \le m \le \frac{k-p-1}{2}. \tag{7}$$

The general term of the sequence, for $1 \le m \le \frac{k-p-1}{2}$, is

$$R_m = \binom{k}{\frac{k+p-1}{2}}^2 \frac{2(m+1)(k-p+1)(k-p-1)}{k(k+p+1)(k+p+3)} \prod_{j=1}^{m} \frac{(k+p+1-2j)(k-p-1-2j)}{(k+p+3+2j)(k-p+1+2j)}. \tag{8}$$

Both R_0 and R_m, for $1 \le m \le \frac{k-p-1}{2} - 1$, are positive and $R_{\frac{k-p-1}{2}}$ is zero. So the result follows from Lemma 9.

Part two. The coefficient of t^{k-p}, for $1 \le p \le k$, is nonnegative.

We can rewrite the coefficient of t^q given in Lemma 12, for $0 \le q \le k-1$, replacing q by $k-p$, for $1 \le p \le k$

$$\sum_{i=0}^{k-p+1} \left[\binom{k}{i}\binom{k}{k-p-i} - \binom{k-1}{i-1}\binom{k+1}{k-p-i+1} \right] c_{k-i}c_{p+i}.$$

The proof depends on the parity of $k-p$.

a) If $k-p$ is even, we now write the coefficient of t^{k-p} grouping together the terms $c_{k-i}c_{p+i}$ and $c_{p+i}c_{k-i}$ as follows

$$\left[\binom{k}{\frac{k-p}{2}}\binom{k}{\frac{k-p}{2}} - \binom{k-1}{\frac{k-p}{2}-1}\binom{k+1}{\frac{k-p}{2}+1} \right] c_{\frac{k+p}{2}} c_{\frac{k+p}{2}} + \sum_{m=1}^{\frac{k-p}{2}+1} r_m c_{\frac{k+p}{2}-m} c_{\frac{k+p}{2}+m}$$

where

$$r_m = 2\binom{k}{\frac{k-p}{2}-m}\binom{k}{\frac{k-p}{2}+m} - \binom{k-1}{\frac{k-p}{2}-m-1}\binom{k+1}{\frac{k-p}{2}+m+1} - \binom{k-1}{\frac{k-p}{2}+m-1}\binom{k+1}{\frac{k-p}{2}-m+1}.$$

We now follow the same procedure as in part one. We build the sequence R_m of the partial sums of the combinatorial coefficients and obtain the multiplicative law of recurrence

$$R_0 = \binom{k}{\frac{k-p}{2}}\binom{k}{\frac{k-p}{2}} - \binom{k-1}{\frac{k-p}{2}-1}\binom{k+1}{\frac{k-p}{2}+1} = \binom{k}{\frac{k-p}{2}}^2 \frac{k+p}{k(k-p+2)}, \tag{9}$$

$$R_m = R_{m-1} \cdot \frac{(2m+1)(k-p-2m+2)(k+p-2m)}{(2m-1)(k-p+2m+2)(k+p+2m)}, \quad \text{for} \quad 1 \le m \le \frac{k-p}{2}. \tag{10}$$

The general term of the sequence, for $1 \le m \le \frac{k-p}{2}$, is

$$R_m = \binom{k}{\frac{k-p}{2}}^2 \frac{(2m+1)(k+p)}{k(k-p+2)} \prod_{j=1}^{m} \frac{(k-p+2-2j)(k+p-2j)}{(k-p+2+2j)(k+p+2j)}. \tag{11}$$

All terms of this sequence, for $1 \leq m \leq \frac{k-p}{2}$, are positive. Now

$$R_{\frac{k-p}{2}+1} = R_{\frac{k-p}{2}} - \binom{k-1}{k-p}\binom{k+1}{0}$$

is the sum of all combinatorial coefficients of t^q, for $0 \leq q \leq k-1$, of $\Delta_k(t)$, *i.e.*

$$\sum_{i=0}^{q+1}\left[\binom{k}{i}\binom{k}{q-i} - \binom{k-1}{i-1}\binom{k+1}{q-i+1}\right] - \binom{k-1}{q}\binom{k+1}{0} = \sum_{i=0}^{q}\binom{k}{i}\binom{k}{q-i} - \sum_{j=0}^{q}\binom{k-1}{j}\binom{k+1}{q-j}$$

and this expression is zero by the formula

$$\sum_{r=0}^{q}\binom{m}{r}\binom{n}{q-r} = \binom{m+n}{q}. \tag{12}$$

Then, the result follows from Lemma 9.

b) If $k-p$ is odd, we now write the coefficient of t^{k-p} grouping together the terms $c_{k-i}c_{p+i}$ and $c_{p+i}c_{k-i}$ as follows

$$\sum_{m=0}^{\frac{k-p-1}{2}+1} r_m c_{\frac{k+p-1}{2}-m} c_{\frac{k+p+1}{2}+m}$$

where

$$r_m = 2\binom{k}{\frac{k-p-1}{2}-m}\binom{k}{\frac{k-p+1}{2}+m} - \binom{k-1}{\frac{k-p-1}{2}-m-1}\binom{k+1}{\frac{k-p+1}{2}+m+1} - \binom{k-1}{\frac{k-p+1}{2}+m-1}\binom{k+1}{\frac{k-p-1}{2}-m+1}.$$

We now follow the same procedure as above. We build the sequence R_m of the partial sums of the combinatorial coefficients and we obtain the multiplicative law of recurrence

$$\begin{aligned} R_0 &= 2\binom{k}{\frac{k-p-1}{2}}\binom{k}{\frac{k-p+1}{2}} - \binom{k-1}{\frac{k-p-1}{2}-1}\binom{k+1}{\frac{k-p+1}{2}+1} - \binom{k-1}{\frac{k-p+1}{2}-1}\binom{k+1}{\frac{k-p-1}{2}+1} \\ &= \binom{k}{\frac{k-p-1}{2}}^2 \frac{2(k+p+1)(k+p-1)}{k(k-p+1)(k-p+3)}, \end{aligned} \tag{13}$$

$$R_m = R_{m-1}.\frac{(m+1)(k-p-2m+1)(k+p-2m-1)}{m(k-p+2m+3)(k+p+2m+1)}, \quad \text{for} \quad 1 \leq m \leq \frac{k-p-1}{2}. \tag{14}$$

The general term of the sequence, for $1 \leq m \leq \frac{k-p-1}{2}$, is

$$R_m = \binom{k}{\frac{k-p-1}{2}}^2 \frac{2(m+1)(k+p+1)(k+p-1)}{k(k-p+1)(k-p+3)} \prod_{j=1}^{m} \frac{(k-p+1-2j)(k+p-1-2j)}{(k-p+3+2j)(k+p+1+2j)}. \tag{15}$$

With arguments similar to the even case, we conclude that all terms of this sequence are positive except that the last is zero. So the result follows from Lemma 9.

Part three. Proof of the multiplicative law of recurrence.

We will prove the law of recurrence in the case $k+p$ even for $0 \leq p \leq k-2$.

We use extensively the relations

$$\binom{k}{s} = \frac{k}{k-s}\binom{k-1}{s}, \quad \binom{k}{s} = \frac{s+1}{k-s}\binom{k}{s+1} \quad \text{and} \quad \binom{k}{s} = \frac{k}{s}\binom{k-1}{s-1}. \tag{16}$$

If $k+p$ is even

$$R_0 = \binom{k}{\frac{k+p}{2}}^2 - \binom{k-1}{\frac{k+p}{2}-1}\binom{k+1}{\frac{k+p}{2}+1} = \binom{k}{\frac{k+p}{2}}^2 - \frac{\frac{k+p}{2}}{k}\binom{k}{\frac{k+p}{2}}\frac{k+1}{\frac{k+p}{2}+1}\binom{k}{\frac{k+p}{2}}$$

$$= \binom{k}{\frac{k+p}{2}}^2\left[1 - \frac{(k+p)(k+1)}{k(k+p+2)}\right] = \binom{k}{\frac{k+p}{2}}^2 \frac{k-p}{k(k+p+2)}.$$

We denote $\frac{k+p}{2} = s$ and then we will prove, by induction, the recurrence given in the expression (4). Rewrite it as

$$R_m = R_{m-1}\cdot\frac{(2m+1)(k-s-m)(s-m+1)}{(2m-1)(k-s+m)(s+m+1)}, \quad \text{for} \quad \frac{k}{2} \leq s \leq k-1. \tag{17}$$

If $m=1$, we use the expression (2) and the formulas (16) to obtain the first term of the recurrence

$$R_1 = R_0 + r_1 = R_0 + 2\binom{k}{s-1}\binom{k}{s+1} - \binom{k-1}{s-2}\binom{k+1}{s+2} - \binom{k-1}{s}\binom{k+1}{s}$$

$$= R_0 + \frac{2s}{k-s+1}\binom{k}{s}\frac{k-s}{s+1}\binom{k}{s} - \frac{(s-1)s}{k(k-s+1)}\binom{k}{s}\frac{(k+1)(k-s)}{(s+2)(s+1)}\binom{k}{s} - \frac{k-s}{k}\binom{k}{s}\frac{k+1}{k-s+1}\binom{k}{s}$$

$$= \binom{k}{s}^2\frac{k-s}{k(s+1)} + \binom{k}{s}^2\left[\frac{2s(k-s)}{(k-s+1)(s+1)} - \frac{(s-1)s(k+1)(k-s)}{k(k-s+1)(s+2)(s+1)} - \frac{(k-s)(k+1)}{k(k-s+1)}\right]$$

$$= \binom{k}{s}^2\frac{k-s}{k(s+1)}\left[1 + \frac{2(sk-k-s^2-s-1)}{(k-s+1)(s+2)}\right] = R_0\frac{3s(k-s-1)}{(k-s+1)(s+2)}.$$

We consider the hypothesis of induction $R_j = R_{j-1}F_j$, for $1 \le j \le m-1$, where F_j is the factor of recurrence

$$F_j = \frac{(2j+1)(s-j+1)(k-s-j)}{(2j-1)(k-s+j)(s+j+1)}$$

and so we have

$$R_{m-1} = R_0 F_1 F_2 \dots F_{m-1}.$$

With this notation, to prove that $R_m = R_{m-1} + r_m = R_{m-1}F_m$, we will see that $R_{m-1}(F_m - 1) = r_m$ or that is equivalent

$$r_m = R_0 F_1 F_2 \dots F_{m-1}(F_m - 1).$$

On the one hand

$R_0 F_1 F_2 \dots F_{m-1}(F_m - 1) =$

$$\binom{k}{s}^2 \frac{k-s}{k(s+1)} \left[\prod_{j=1}^{m-1} \frac{(2j+1)(s-j+1)(k-s-j)}{(2j-1)(k-s+j)(s+j+1)} \right] \left[\frac{(2m+1)(s-m+1)(k-s-m)}{(2m-1)(k-s+m)(s+m+1)} - 1 \right].$$

In order to simplify the following expressions, we denote

$$B = \binom{k}{s} \prod_{j=1}^{m-1} \frac{(s-j+1)}{(k-s+j)} \quad \text{and} \quad D = \binom{k}{s} \prod_{j=1}^{m-1} \frac{(k-s-j)}{(s+j+1)}$$

and then we have

$$R_0 F_1 F_2 \dots F_{m-1}(F_m - 1) = \frac{k-s}{k(s+1)} \left[\frac{-2(s^2 + s(1-k) + k(2m^2-1) + m^2}{(k-s+m)(s+m+1)} \right] BD.$$

On the other hand, sustituting $\frac{(k+p)}{2} = s$ in (2), we have

$$r_m = 2\binom{k}{s-m}\binom{k}{s+m} - \binom{k-1}{s-m-1}\binom{k+1}{s+m+1} - \binom{k-1}{s+m-1}\binom{k+1}{s-m+1}.$$

We now use the formulas (16) and the above notations to obtain

$$r_m = 2\frac{(s-m+1)}{(k-s+m)}B\frac{(k-s)}{(s+1)}D - \frac{(s-m+1)(s-m)}{k(k-s+m)}B\frac{(k+1)(k-s)}{(s+1)(s+m+1)}D$$
$$-\frac{(k-s)(s+m)}{k(s+1)}D\frac{(k+1)}{(k-s+m)}B.$$

Finally, it is a routine exercise to verify that the last expressions of $R_0 F_1 F_2 \dots F_{m-1}(F_m - 1)$ and r_m coincide.

The proofs of the laws of recurrence in the cases $k+p$ odd, for $0 \leq p \leq k-2$, and the cases $k-p$ even and odd, for $1 \leq p \leq k$, are similar to the above. ■

Corollary 14. *If c is a Newton sequence satisfying $c_k \geq 0$ for even k and $c_k \leq 0$ for odd k, then $c(t)$ is a Newton sequence for all $t \leq 0$.*

5 Eventually Newton Sequences

We call a sequence c **right** (resp. **left**) **eventually Newton** if there is a $T \in \mathbb{R}$ such that $c(t)$ is a Newton sequence for all $t > T$ (resp. for all $t < T$). If at least one of the two occurs, the sequence is simply called **eventually Newton** (e-Newton). If for a sequence c no translation $c(t)$ is a Newton sequence, then c is called **never Newton** (n-Newton). Our purpose in this section is to show that every sequence is either eventually or never Newton (justifying the language). We do this by determining which sequences are eventually Newton. Because of Lemma 12, a key is the first Newton difference Δ_1. In the matrix context, this has an interesting interpretation. If $A = (a_{ij})$, then

$$n(n-1)\Delta_1 = \frac{1}{n}\sum_{i<j}(a_{ii}-a_{jj})^2 + 2\sum_{i<j} a_{ij}a_{ji}. \tag{18}$$

Theorem 15. *Let $c : c_0 = 1, c_1, \ldots, c_n$ be a sequence.*

(a) *If $\Delta_1 > 0$, then c is right eventually Newton.*

(b) *If $\Delta_1 = 0$ and $\Delta_2 \neq 0$, then c is right eventually Newton when $\Delta_2/c_1 > 0$ and left eventually Newton when $\Delta_2/c_1 < 0$.*

(c) *If $\Delta_1 = \Delta_2 = \cdots = \Delta_r = 0$ and $\Delta_{r+1} \neq 0$, with $r \geq 2$, then c is:*

  (c1) *e-Newton if r is even and $c_1 = 0$.*

  (c2) *right eventually Newton if r is even and $c_1 \neq 0$.*

  (c3) *right eventually Newton if r is odd and $c_1 = 0$.*

  (c4) *e-Newton if r is odd and $c_1 \neq 0$.*

Proof: In Lemma 12 we obtained the polynomial $\Delta_k(t)$, for $k = 1, \ldots, n-1$:

$$\sum_{q=0}^{2k-2}\left(\sum_{j=q-k}^{k}\left[\binom{k}{j}\binom{k}{q-j} - \binom{k-1}{j-1}\binom{k+1}{q+1-j}\right]c_{k-j}c_{k-(q-j)}\right)t^q =$$

$$(c_1^2 - c_0c_2)t^{2k-2} + (k-1)(c_1c_2 - c_0c_3)t^{2k-3} + (k-1)(\frac{k}{2}c_2c_2 - c_1c_3 - \frac{k-2}{2}c_0c_4)t^{2k-4} + \cdots$$

Note that

$$\sum_{j=q-k}^{k}\left[\binom{k}{j}\binom{k}{q-j} - \binom{k-1}{j-1}\binom{k+1}{q+1-j}\right] = \sum_{j=0}^{q}\left[\binom{k}{j}\binom{k}{q-j} - \binom{k-1}{j-1}\binom{k+1}{q+1-j}\right]$$

because

$$\binom{k}{j} = \binom{k-1}{j-1} = 0 \quad \text{if} \qquad \mathrm{q \le k-1 \quad and \quad j < 0 \qquad or \qquad q > k \quad and \quad j > k}$$

and

$$\binom{k}{q-j} = \binom{k+1}{q+1-j} = 0 \quad \text{if} \qquad \mathrm{q \le k-1 \quad and \quad j > q \qquad or \qquad q > k \quad and \quad j < q-k.}$$

On the one hand, we can use the formula

$$\sum_{r=0}^{q} \binom{m}{r}\binom{n}{q-r} = \binom{m+n}{q}$$

to prove that the sum of the combinatorial coefficients of each term in t^q of the polynomial $\Delta_k(t)$ is zero. On the other hand, we have observed in Section 1 that

$$\Delta_1 = \cdots = \Delta_r = 0 \quad \Longleftrightarrow \quad c_k = \begin{cases} c_1^k & k = 1, \ldots, r+1 \quad \text{if} \quad c_1 \neq 0 \\ 0 & k = 2, \ldots, r \qquad\quad \text{if} \quad c_1 = 0. \end{cases}$$

To prove the theorem it is sufficient to consider the dominant term of the polynomial $\Delta_k(t)$:

(a) If $\Delta_1 = c_1^2 - c_0 c_2 > 0$ then $\Delta_k(t) = (c_1^2 - c_0 c_2)t^{2k-2} + \cdots$ and it is obvious that c is right eventually Newton.

(b) If $\Delta_1 = c_1^2 - c_0 c_2 = 0$ then $c_2 = c_1^2$ and $\Delta_2 = c_1(c_1^3 - c_3) \neq 0$. Now $\Delta_k(t) = (k-1)(c_1^3 - c_3)t^{2k-3} + \cdots$ and so c is right eventually Newton if $c_1^3 - c_3 = \frac{\Delta_2}{c_1} > 0$ and c is left eventually Newton if $c_1^3 - c_3 = \frac{\Delta_2}{c_1} < 0$.

(c) If $c_1 = 0$ the polynomial has the form

$$\Delta_k(t) = \frac{(k-1)(k-2)\cdots(k-(r-1))}{(r-1)!}(-c_{r+1})t^{2k-(r+1)} + \cdots$$

so (c1) and (c3) are clear and depend of the sign of c_{r+1}.

If $c_1 \neq 0$ the polynomial is

$$\Delta_k(t) = \frac{(k-1)(k-2)\cdots(k-r)}{r!}(c_1^{r+2} - c_{r+2})t^{2k-(r+2)} + \cdots$$

so (c2) and (c4) are clear and depend of the sign of $c_1^{r+2} - c_{r+2} = \frac{\Delta_{r+1}}{c_1^r} > 0$. ■

It now follows from Theorem 15 that all sequences are either e-Newton or n-Newton and that all sequences have been classified as one or the other. It follows from (18), as well as Lemma 12, that $\Delta_1(t)$, equal to Δ_1, is constant; so if, for a sequence c, $\Delta_1 < 0$, then for the sequence $c(t)$, $\Delta_1(t) < 0$ for all real t, and such a sequence is n-Newton. If $0 = \Delta_1 = \Delta_2 = \cdots = \Delta_{n-1}$, then the sequence c is not only Newton, but f-Newton:

If $\mathrm{c} : 1, 0, \ldots, 0, \mathrm{c_n} \quad \Rightarrow \quad c(t) = c \quad \Rightarrow \quad \Delta_k(t) = \Delta_k = 0, \; 1 \le k \le n-1.$

If $\mathrm{c} : 1, \mathrm{c_1}, \mathrm{c_1^2}, \ldots, \mathrm{c_1^n} \quad \Rightarrow \quad c(t) = 1, c_1 + t, (c_1+t)^2, \ldots, (c_1+t)^n \quad \Rightarrow \quad \Delta_k(t) = \Delta_k = 0, \; 1 \le k \le n-1.$

All other sequences are classified within Theorem 15.

References

[1] James, G., Johnson, C.R., Pierce, S., Generalized matrix function inequalities on M-matrices, J. London. Math. Soc. (2) 57 (1998) n.3 562-582.

[2] Holtz, O., M-matrices satisfy Newton's inequalities, Proc. Amer. Math. Soc. 133 (2005) n.3 711-717.

[3] Maclaurin, C., A second letter to Martin Folkes, Esq.; concerning the roots of equations, with the demostration of other rules in algebra, Phil. Transactions 36 (1729) 59-96.

[4] Newton, I., Arithmetica universalis: sive de compositione et resolutione arithmetica liber, 1707.

[5] Niculescu, C. P., A new look at Newton's inequalities, J. Inequal. Pure Appl. Math. 1(2000) no. 2, Article 17, 14 pp. (electronic).