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The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. EBL digraphs

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Abstract

The nonnegative inverse eigenvalue problem (NIEP) is: given a family of complex numbers $\sigma = \{\lambda_1, \ldots, \lambda_n\}$, find necessary and sufficient conditions for the existence of a nonnegative matrix A of order n with spectrum σ. Loewy and London [R. Loewy, D. London, A note on the inverse eigenvalue problems for nonnegative matrices, Linear and Multilinear Algebra 6 (1978) 83–90] resolved it for $n = 3$, and for $n = 4$ when the spectrum is real. In our way of handling the NIEP, we focus our attention on the coefficients of the characteristic polynomial of A. Thus, the NIEP that we consider is: "*given* $k_1, k_2, \ldots, k_n$ *real numbers, find necessary and sufficient conditions for the existence of a nonnegative matrix A of order n with characteristic polynomial* $x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_n$". The coefficients of the characteristic polynomial are closely related to the cyclic structure of the weighted digraph with adjacency matrix A. We introduce a special type of digraph structure, that we shall call EBL, in which this relation is specially simple. We give some results that show the interest of EBL structures. We completely solve the NIEP from

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the coefficients of the characteristic polynomial for $n = 4$. We also solve a special case of the NIEP for $n \leqslant 2p + 1$ with $k_1 = \cdots = k_{p-1} = 0$ and $p \geqslant 2$.

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1. Introduction

First we consider the nonnegative inverse eigenvalue problem (NIEP): given a family of complex numbers $\sigma = \{\lambda_1, \ldots, \lambda_n\}$, find necessary and sufficient conditions for the existence of a nonnegative matrix A of order n with spectrum σ. As Johnson [8] said "This is an intriguing and difficult problem, the resolution to which appears to be far from known".

Necessary conditions for σ to be the spectrum of a nonnegative matrix A of order n are:

(C1) σ is closed under the complex conjugation;
(C2) the spectral radius ρ of A is in σ;
(C3) the moments of all the orders are nonnegative;
where the **moment** of order k of σ is the number

$$s_k(\sigma) := \sum_{i=1}^{n} \lambda_i^k = \operatorname{tr} A^k, \quad k = 1, 2, \ldots \tag{1}$$

Loewy and London [12] in 1978, and Johnson [8] independently in 1979, put forward another transcendental necessary condition for studying the NIEP:

(C4) $(s_k(\sigma))^m \leqslant n^{m-1} s_{km}(\sigma), \quad k, m = 1, 2, \ldots$

It is well known, Friedland [6], that the condition (C3) implies that the spectral radius of A is in σ. Loewy and London [12] use the Newton identities

$$s_m + k_1 s_{m-1} + k_2 s_{m-2} + \cdots + k_{m-1} s_1 + k_m m = 0 \quad m = 1, \ldots, n \tag{2}$$

that relate the coefficients of the characteristic polynomial

$$\prod_{j=1}^{n} (x - \lambda_j) = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_n \tag{3}$$

with the moments s_k of the eigenvalues to show that (C3) implies that σ is closed under the complex conjugation. Consequently, the necessary conditions in the NIEP can be reduced to (C3) and (C4). Recently, Holtz [7] gave new necessary conditions, *Newton's Inequalities*:

(C5) $(c_i(\sigma))^2 \geqslant c_{i-1}(\sigma) c_{i+1}(\sigma), i = 1, \ldots, n-1,$
where

$$c_i(\sigma) = \frac{\text{coefficient of degree } (n-i) \text{ of } \prod_{j=1}^{n}(x - (\rho - \lambda_j))}{\binom{n}{i}}. \tag{4}$$

Holtz proves that the conditions (C3) and (C4) and (C5) are mutually independent.

On the other hand, Suleimanova, Brauer and Perfect (1949–1955) introduced seminal geometric and algebraic techniques to deal with this problem. These techniques have provided the basis for dozens of articles over the last 50 years that have proposed many sufficient conditions with weak mutual implications. Most such conditions only consider the case where the spectrum is real. Those given by Kellogg [9] in 1971 and Salzman [17] in 1972 stand out. In the last 10 years, the necessary condition of Johnson–Loewy–London has been efficiently exploited to advance the solution to the NIEP in very particular cases, with sufficient conditions that can be expressed by means of relations between moments of different orders. Thus, Reams [15] in 1996 resolved the NIEP for matrices of order 4 and zero trace and gave a sufficient condition for matrices of order 5 and zero trace. Later, Laffey and Meehan [11] in 1999 resolved the problem for matrices of order 5 and zero trace. Borobia [2] improved Kellogg's condition in 1995 and Soto [18] in 2003 generalized Salzman's sufficient condition. Rojo and Soto [16] and Borobia, Moro and Soto [3,19] have made the most recent contributions to the NIEP.

However, these sufficient conditions seem to be far from the known necessary conditions. If complete characterizations of this problem are to be looked for, little is known. The NIEP is trivial for $n \leqslant 2$. In 1978, Loewy and London [12] resolved it for $n = 3$ (see our Section 4), and for $n = 4$ in the particular case where the spectrum is real. The general case for $n \geqslant 4$, at present, remains open.[3]

Other significant contributions related to the NIEP: in 1991 Boyle and Handelman [4] studied the families of nonzero complex numbers, which are the nonzero portion of the spectrum of a nonnegative matrix. They characterized the nonzero spectra of primitive matrices using symbolic dynamics. A problem, which remains open, is to find the minimum number of zeros to add, or failing that, a good lower bound. In 1997, Wuwen [20] set bounds to the minimum value of the spectral radius of a collection of complex numbers, that is closed under the complex conjugation, realizable as the spectrum of a nonnegative matrix; this minimum remains to be found.

In our way of handling the NIEP, a nonnegative matrix will be seen as the adjacency matrix of a weighted digraph. We shall not focus our attention directly on its spectrum but on the coefficients of its characteristic polynomial. Thus, the NIEP that we consider can be described as follows:

> "*given real numbers $k_1, k_2, \dots, k_n$, find necessary and sufficient conditions for the existence of a nonnegative matrix of order n with characteristic polynomial $x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_n$*".

We shall say that such a polynomial $P(x)$ is **realizable** and that the nonnegative matrix with characteristic polynomial $P(x)$ is a **matricial realization** of the polynomial. The coefficients of the characteristic polynomial are closely related to the cyclic structure of the weighted digraph associated to the matrix A, as established by the Coefficient Theorem (see Section 2). Our purpose is to introduce tools that allow us to relate the information contained in the cyclic structure of the digraph associated with A to the coefficients of its characteristic polynomial. To achieve this we shall introduce a special type of digraph structure, that we shall call EBL, in which the desired connections are specially simple. This relation between the cyclic structure and the coefficients of the characteristic polynomial, which is workable thanks to the EBL digraphs, is the basis of the results obtained that, for $n = 4$, completely resolve the NIEP.

[3] Added in proof: It was pointed out to us by the referee that the NIEP for $n = 4$ has been studied by Meehan in her Ph.D. thesis [13] with a different approach from the one in the present paper.

The coefficients of the characteristic polynomial have been taken into consideration very little in the context of the NIEP. Apart from its use as a proof tool already mentioned in [12], Perfect and Mirsky [14] in 1965 characterized the polynomials of degree three that are characteristic polynomials of doubly stochastic matrices.

The rest of this paper is organized as follows:

In Section 2, we introduce basic concepts, notations and results used in this paper.

In Section 3, we give some necessary conditions for the nonnegative matricial realization of a polynomial of degree n. In particular, in Theorem 3, necessary and sufficient conditions are given for the coefficients k_1, k_2 and k_3 so that a polynomial of the form $x^n + k_1x^{n-1} + k_2x^{n-2} + k_3x^{n-3} + \cdots$ can be realizable.

In Section 4, we specify the solution in the cases $n = 2$ and $n = 3$.

In Section 5, we introduce the EBL digraphs and matrices. We give two general results that show the interest of these structures and we give an explicit procedure to transform, in the cases $n = 3$ and $n = 4$, any matricial realization into an EBL realization.

In Section 6, we solve the NIEP from the coefficients of the characteristic polynomial for the case $n = 4$.

In Section 7, we solve the NIEP in the case $n \leqslant 2p + 1$ with $k_1 = \cdots = k_{p-1} = 0$ and $p \geqslant 2$, which includes the case $n = 5$ when $k_1 = 0$ solved by Laffey and Meehan [11].

2. Preliminaries and notations

In this paper we will use some standard basic concepts and results about square nonnegative matrices such as **reducible**, **irreducible**, **Frobenius normal form** of a reducible matrix, **irreducible component** and **Frobenius Theorem** about the spectral structure of an irreducible matrix as they have been described in [1].

By a **weighted digraph** G, or simply **digraph**, we mean a triplet (V, E, w) where V is a nonempty finite set, $E \subset V \times V$ and $w: E \to \mathbb{R}^+$ is a positive real map on E. The elements in V and E are called **vertices** and **arcs** respectively; the values of the map w are called **weights**. The **adjacency matrix** of a weighted digraph (V, E, w) with $V = \{v_1, \ldots, v_n\}$ is the matrix $A = (a_{ij})_{i,j=1}^n$ where $a_{ij} = w(v_i, v_j)$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$ otherwise.

A sequence of different vertices $v_1v_2 \cdots v_r$, $r \geqslant 1$, such that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \ldots, r - 1$ is called a **path** of **length** $r - 1$ joining v_1 with v_r. A **cycle** of **length** r or r**-cycle** is a sequence of vertices $v_1v_2 \cdots v_rv_1$ where $v_1v_2 \cdots v_r$ is a path and $(v_r, v_1) \in E$. A **linear digraph** is a collection of disjoint cycles. A digraph is **strongly connected** if every two vertices are joined by a path.

A **subdigraph** of (V, E, w) is a digraph (V', E', w') with $V' \subset V$, $E' \subset E$ and $w' = w_{|E'}$. The subdigraph will be called an **induced subdigraph** when $E' = E \cap (V' \times V')$.

Coefficient Theorem for weighted digraphs: *Let G be a weighted digraph, A its adjacency matrix and $P_G(x) = P_A(x) = |xI - A| = x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_n$. Then, for each $1 \leqslant i \leqslant n$,*

$$k_i = \sum_{L \in \mathscr{L}_i} (-1)^{p(L)} \Pi(L), \tag{5}$$

where $\mathscr{L}_i$ is the set of all linear subdigraphs L of G with exactly i vertices; $p(L)$ denotes the number of cycles of L; $\Pi(L)$ denotes the product of the weights of all arcs belonging to L. (See [5]).

Let $A = (a_{ij})_{i,j=1}^n$ be the adjacency matrix of a weighted digraph G. For $r \geqslant 1$, we denote as $c_{i_1 i_2 \dots i_r}$ the weight of the r-cycle joining the vertices $v_{i_1}, v_{i_2}, \dots, v_{i_r}$, that is

$$c_{i_1 i_2 \dots i_r} = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_r i_1} (= c_{i_r i_1 i_2 \dots i_{r-1}} = c_{i_{r-1} i_r i_1 i_2 \dots i_{r-2}} = \cdots = c_{i_2 i_3 \dots i_r i_1}). \tag{6}$$

When $r = 1$ we will put

$$l_i = a_{ii} = c_i, \tag{7}$$

that is the weight of the 1-cycle or **loop** at vertex v_i. Let $1 \leqslant i_1 \leqslant \cdots \leqslant i_q \leqslant n$ be a sequence of integers. We denote as $CS_{i_1 \dots i_q}$ the subset of $\mathscr{L}_{i_1 + \dots + i_q}$ whose elements are sets of q disjoint cycles of G of lengths $i_1, \dots, i_q$; CS from *cyclic structures*. Finally, let $f_m(l_1, \dots, l_n)$ be the symmetric function on $l_1, \dots, l_n$, that is

$$f_m(l_1, \dots, l_n) = \begin{cases} \sum\limits_{i_1 < \dots < i_m} l_{i_1} \cdots l_{i_m} & \text{if } 1 \leqslant m \leqslant n, \\ 0 & \text{if } m > n. \end{cases} \tag{8}$$

Hence, for $P_G(x) = P_A(x) = x^n + k_1 x^{n-1} + \cdots + k_n$, we have

$$k_1 = -\sum_{CS_1} l_i = -f_1(l_1, \dots, l_n), \tag{9}$$

$$k_2 = \sum_{CS_{11}} l_i l_j - \sum_{CS_2} c_{ij} = f_2(l_1, \dots, l_n) - \sum_{CS_2} c_{ij}, \tag{10}$$

$$\begin{aligned} k_3 &= -\sum_{CS_{111}} l_i l_j l_r + \sum_{CS_{12}} l_i c_{jr} - \sum_{CS_3} c_{ijr} \\ &= -f_3(l_1, \dots, l_n) + \sum_{CS_{12}} l_i c_{jr} - \sum_{CS_3} c_{ijr}. \end{aligned} \tag{11}$$

3. Necessary conditions

Proposition 1. *Let* $P(x) = k_0 x^n + k_1 x^{n-1} + \cdots + k_n$ *be a polynomial with real coefficients,* $n \geqslant 1$ *and* $k_0 > 0$. *Then,* $\forall x > \max\{\operatorname{Re} \lambda : P(\lambda) = 0\}$, $P^{(j)}(x) > 0$, *for* $j = 0, 1, \dots, n$.

Proof. The result is clear for $n = 1, 2$. When $n > 2$, it can be proved by induction over n writing $P(x) = P_1(x) P_2(x)$, with $P_1(x)$ and $P_2(x)$ polynomials verifying the hypothesis of the proposition and with degree lower than n. Now the Leibniz formula for the derivative gives the result. □

Corollary 2. *Let* $P(x)$ *be the characteristic polynomial of a nonnegative matrix of order n with spectral radius* ρ. *Then,* $\forall x > \rho$, $P^{(j)}(x) > 0$, *for* $j = 0, 1, \dots, n$.

Theorem 3. *Let* $P(x) = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_n$ *be the characteristic polynomial, of degree* $n \geqslant 3$, *of a nonnegative matrix* A. *Then*:

(a) $k_1 \leqslant 0$; (12)

(b) $k_2 \leqslant \dfrac{n-1}{2n} k_1^2$; (13)

$$\text{(c)}\ \ k_3 \leqslant \begin{cases} \frac{n-2}{n}\left(k_1k_2+\frac{n-1}{3n}\left(\left(k_1^2-\frac{2nk_2}{n-1}\right)^{\frac{3}{2}}-k_1^3\right)\right) & \text{if } \frac{(n-1)(n-4)}{2(n-2)^2}k_1^2<k_2, \\ k_1k_2-\frac{(n-1)(n-3)}{3(n-2)^2}k_1^3 & \text{if } k_2 \leqslant \frac{(n-1)(n-4)}{2(n-2)^2}k_1^2. \end{cases} \tag{14}$$

Moreover, given k_1, k_2 and k_3 verifying the above conditions there exists a nonnegative matrix of order n whose characteristic polynomial is of the form $x^n+k_1x^{n-1}+k_2x^{n-2}+k_3x^{n-3}+Q(x)$, where $Q(x)=0$ if $n=3$ and a polynomial of degree lower than or equal to $n-4$ if $n>3$.

Proof. (a) $k_1=-\mathrm{tr}(A)\leqslant 0$ because A is a nonnegative matrix.

(b) Using the Coefficient Theorem and (10), for a fixed k_1, the maximum value of k_2 is obtained when there are no 2-cycles and the weight of the loops is equally distributed, that is $c_{ij}=0$, for $i\neq j$, and $l_1=l_2=\cdots=l_n=-\frac{k_1}{n}$. Then

$$k_2\leqslant \binom{n}{2}\left(-\frac{k_1}{n}\right)^2=\frac{n-1}{2n}k_1^2. \tag{15}$$

(c) Again, using the Coefficient Theorem and the expressions (9)–(11), for fixed k_1 and k_2, the maximum value of k_3 is obtained when there are no 3-cycles and the weight of all 2-cycles is focussed on 2-cycles connecting two vertices with loops of lowest weight. Without loss of generality we can assume $l_1\leqslant l_2\leqslant\cdots\leqslant l_n$, and so we can take $c_{ij}=0$ for $(i,j)\neq(1,2)$. Let

$$s=\sum_{i\geqslant 3} l_i \tag{16}$$

and note that

$$-\frac{n-2}{n}k_1\leqslant s\leqslant -k_1. \tag{17}$$

Then we have

$$\begin{aligned} k_1&=-f_1(l_1,\ldots,l_n)=-(l_1+l_2)-s,\\ k_2&=f_2(l_1,\ldots,l_n)-c_{12},\\ k_3&=-f_3(l_1,\ldots,l_n)+c_{12}\sum_{i\geqslant 3}l_i=-f_3(l_1,\ldots,l_n)+c_{12}s. \end{aligned} \tag{18}$$

From the above expressions of k_1 and k_2 we obtain:

$$\begin{aligned} l_1+l_2&=-k_1-s,\\ c_{12}&=f_2(l_1,\ldots,l_n)-k_2. \end{aligned} \tag{19}$$

This allows us to express k_3 as:

$$\begin{aligned} k_3&=-(l_1l_2s+(l_1+l_2)f_2(l_3,\ldots,l_n)+f_3(l_3,\ldots,l_n))+c_{12}s\\ &=-(l_1l_2s+(-k_1-s)f_2(l_3,\ldots,l_n)+f_3(l_3,\ldots,l_n))+(f_2(l_1,\ldots,l_n)-k_2)s. \end{aligned} \tag{20}$$

Because

$$\begin{aligned} f_2(l_1,\ldots,l_n)&=l_1l_2+(l_1+l_2)s+f_2(l_3,\ldots,l_n)\\ &=l_1l_2-(k_1+s)s+f_2(l_3,\ldots,l_n) \end{aligned} \tag{21}$$

we have the following expression:

$$k_3 = -s^3 - k_1 s^2 - k_2 s + (k_1 + 2s) f_2(l_3, \dots, l_n) - f_3(l_3, \dots, l_n). \tag{22}$$

Let us now see that for $s \in \left[-\frac{n-2}{n}k_1, -k_1\right]$, see (17), the function

$$H^{[s]}(l_3, \dots, l_n) = (k_1 + 2s) f_2(l_3, \dots, l_n) - f_3(l_3, \dots, l_n) \tag{23}$$

attains its maximum on the set $B = \{(l_3, \dots, l_n)/0 \leqslant l_3 \leqslant l_4 \leqslant \dots \leqslant l_n, l_3 + l_4 + \dots + l_n = s\}$ at

$$l_3 = l_4 = \dots = l_n = \frac{s}{n-2}. \tag{24}$$

The maximum exists because the function $H^{[s]}$ is continuous and B is a compact set. Let us assume this maximum is attained at a point $(l_3, \dots, l_n)$ with $l_i < l_{i+1}$, for some $i < n$. Put $\tilde{l}_i = \tilde{l}_{i+1} = (l_i + l_{i+1})/2$, then

$$\begin{aligned} &H^{[s]}(l_3, \dots, l_n) - H^{[s]}(l_3, \dots l_{i-1}, \tilde{l}_i, \tilde{l}_{i+1}, l_{i+2}, \dots, l_n) \\ &\quad = (l_i l_{i+1} - \tilde{l}_i \tilde{l}_{i+1})(-l_1 - l_2 + l_i + l_{i+1}) < 0 \end{aligned} \tag{25}$$

which contradicts the assumed maximum.

Now, if we replace $l_3, l_4, \dots, l_n$ by $s/(n-2)$ in the expression of k_3 obtained in (22) we have:

$$\begin{aligned} k_3 &= -s^3 - k_1 s^2 - k_2 s + (k_1 + 2s)\binom{n-2}{2}\left(\frac{s}{n-2}\right)^2 - \binom{n-2}{3}\left(\frac{s}{n-2}\right)^3 \\ &= -\frac{n(n-1)}{3!(n-2)^2}s^3 - \frac{n-1}{2(n-2)}k_1 s^2 - k_2 s \\ &\leqslant \max_{-\frac{n-2}{n}k_1 \leqslant s \leqslant -k_1} \left\{-\frac{n(n-1)}{3!(n-2)^2}s^3 - \frac{n-1}{2(n-2)}k_1 s^2 - k_2 s\right\} \\ &= \max_{-\frac{k_1}{n} \leqslant l_n \leqslant -\frac{k_1}{n-2}} \left\{-\frac{n(n-1)(n-2)}{3!}l_n^3 - \frac{(n-1)(n-2)}{2}k_1 l_n^2 - (n-2)k_2 l_n\right\}. \end{aligned} \tag{26}$$

Let

$$k_3^{\max}(k_1, k_2) = \max_{-\frac{k_1}{n} \leqslant l_n \leqslant -\frac{k_1}{n-2}} \left\{-\frac{n(n-1)(n-2)}{3!}l_n^3 - \frac{(n-1)(n-2)}{2}k_1 l_n^2 - (n-2)k_2 l_n\right\} \tag{27}$$

and let $l_n^{k_3^{\max}}$ be the value of l_n where $k_3^{\max}(k_1, k_2)$ is attained. Then we have

$$l_n^{k_3^{\max}}(k_1, k_2) = \begin{cases} -\frac{k_1}{n} + \frac{1}{n}\sqrt{k_1^2 - \frac{2nk_2}{n-1}} & \text{if } \frac{(n-1)(n-4)}{2(n-2)^2}k_1^2 < k_2, \\ -\frac{k_1}{n-2} & \text{if } k_2 \leqslant \frac{(n-1)(n-4)}{2(n-2)^2}k_1^2 \end{cases} \tag{28}$$

and

$$k_3^{\max}(k_1,k_2)=\begin{cases}\frac{n-2}{n}\left(k_1k_2+\frac{n-1}{3n}\left(\left(k_1^2-\frac{2nk_2}{n-1}\right)^{\frac{3}{2}}-k_1^3\right)\right) & \text{if } \frac{(n-1)(n-4)}{2(n-2)^2}k_1^2<k_2,\\ k_1k_2-\frac{(n-1)(n-3)}{3(n-2)^2}k_1^3 & \text{if } k_2\leqslant\frac{(n-1)(n-4)}{2(n-2)^2}k_1^2,\end{cases} \tag{29}$$

which proves condition (c).

Finally, given k_1, k_2 and k_3 verifying (a)–(c), the nonnegative matrix

$$\begin{pmatrix} l_1 & 1 & 0 & \cdots & \cdots & 0\\ c_{12} & l_1 & 1 & 0 & \cdots & 0\\ c_{123} & 0 & l_n & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & \ddots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & 1\\ 0 & 0 & 0 & \cdots & 0 & l_n\end{pmatrix} \quad\text{where}\quad \begin{cases} l_n=l_n^{k_3^{\max}}(k_1,k_2),\\ l_1=\frac{-k_1-(n-2)l_n}{2},\\ c_{12}=f_2(l_1,l_1,l_n,\ldots,l_n)-k_2,\\ c_{123}=k_3^{\max}(k_1,k_2)-k_3,\end{cases} \tag{30}$$

has its characteristic polynomial of the form $x^n+k_1x^{n-1}+k_2x^{n-2}+k_3x^{n-3}+Q(x)$. □

Let us assume that the polynomial $P(x)=x^n+k_1x^{n-1}+\cdots+k_jx^{n-j}+\cdots+k_n$ is realizable. A frequent objective throughout this paper will be to try to maximize the coefficient of degree $n-j$ as a function of the coefficients of higher degree maintaining the realizability for a polynomial of degree n with equal $k_1,\ldots,k_{j-1}$ as $P(x)$. This maximum, which is attained in the cases considered, will be denoted by $k_j^{\max}(k_1,\ldots,k_{j-1})$. Note that this maximum expression depends on the degree n of the polynomial. The proof of the previous theorem clearly shows this dependency on n: see (29) for $k_3^{\max}(k_1,k_2)$.

Proposition 4. *Let $P(x)=x^n+k_1x^{n-1}+k_2x^{n-2}+\cdots+k_n$ be the characteristic polynomial of a nonnegative matrix with spectral radius ρ such that $P(\rho)=P(-\rho)=0$ and $|P'(-\rho)|>P'(\rho)$. Then there exists $\varepsilon_0>0$ such that $P(x)+\varepsilon$ is not realizable for $0<\varepsilon\leqslant\varepsilon_0$.*

Proof. We know, from Corollary 2, that $P'(\rho)\geqslant 0$. If $P'(\rho)=0$, the result is true (see Corollary 2). When $P'(\rho)>0$, in some neighbourhoods of $-\rho$ and ρ we have $|P'(x)|>P'(y)>0$. Now, this combined with the Mean Value Theorem gives the result. □

4. The cases $n=2$ and $n=3$

Theorem 3 can be extended to the case $n=2$.

Theorem 5. *Let $P(x)=x^2+k_1x+k_2$. Then the following two statements are equivalent:*

(i) *$P(x)$ is the characteristic polynomial of a nonnegative matrix;*
(ii) *the coefficients of $P(x)$ satisfy the following conditions:*

(a) $k_1\leqslant 0,$ (31)

(b) $k_2\leqslant\dfrac{k_1^2}{4}.$ (32)

Further, when (i) *and* (ii) *hold, a matricial realization for* $P(x)$ *is*

$$\begin{pmatrix} l_1 & 1 \\ c_{12} & l_1 \end{pmatrix} \quad \textit{where} \quad \begin{cases} l_1 = -\frac{k_1}{2}, \\ c_{12} = \frac{k_1^2}{4} - k_2. \end{cases} \tag{33}$$

Theorem 3 for the case $n = 3$ provides the following characterization:

Theorem 6. *Let* $P(x) = x^3 + k_1x^2 + k_2x + k_3$. *Then the following two statements are equivalent*:

(i) $P(x)$ *is the characteristic polynomial of a nonnegative matrix*;
(ii) *the coefficients of* $P(x)$ *verify the following conditions*:

(a) $k_1 \leqslant 0,$ (34)

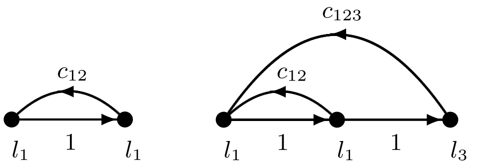
(b) $k_2 \leqslant \dfrac{k_1^2}{3},$ (35)

(c) $k_3 \leqslant k_3^{\max}(k_1, k_2) = \begin{cases} \frac{k_1k_2}{3} + \frac{2}{27}\left((k_1^2 - 3k_2)^{\frac{3}{2}} - k_1^3\right) & \textit{if } k_2 > -k_1^2, \\ k_1k_2 & \textit{if } k_2 \leqslant -k_1^2. \end{cases}$ (36)

Moreover, when (i) *and* (ii) *hold, a matricial realization for* $P(x)$ *is*

$$\begin{pmatrix} l_1 & 1 & 0 \\ c_{12} & l_1 & 1 \\ c_{123} & 0 & l_3 \end{pmatrix} \quad \textit{where} \quad \begin{cases} l_3 = l_3^{k_3^{\max}}(k_1, k_2), \\ l_1 = \frac{-k_1 - l_3}{2}, \\ c_{12} = f_2(l_1, l_1, l_3) - k_2, \\ c_{123} = k_3^{\max}(k_1, k_2) - k_3. \end{cases} \tag{37}$$

Remark 7. Observe that the digraphs associated to the realizations given in the previous theorems can be represented as



Remark 8. When $n \leqslant 3$ the graph of the polynomial function P tells us if the polynomial is realizable or not.

The graph of $P(x) = x^3 + k_1x^2 + k_2x + k_3$ has an inflexion point at $x = -k_1/3$ and $P'(-k_1/3) = k_2 - k_1^2/3$. Suppose $P(x)$ is realizable, then:

– Condition (a) says that the x-coordinate of the inflexion point $-k_1/3$ is in the interval $[0, +\infty)$.
– Condition (b) says that $P'(-k_1/3) = k_2 - k_1^2/3 \leqslant 0$ (we have a horizontal tangent at the inflexion point when k_2 is maximum and the slope of this tangent decreases with the distance of k_2 to $k_1^2/3$).
– Condition (c) says that the graph of $P(x)$ is obtained by pulling down the graph of $x^3 + k_1x^2 + k_2x + k_3^{\max}(k_1, k_2)$ via a vertical translation. Reciprocally, if the graph of a realizable polynomial is moved up, we get a realizable polynomial until its spectral radius ρ either

becomes a multiple root or ρ and $-\rho$ become roots. Above these situations we will go against Corollary 2 or Proposition 4, respectively.

Loewy and London, see [12], proved that given a family $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ of complex numbers, the following conditions

$$\text{(L1)} \quad \max_{1\leqslant i\leqslant 3} |\lambda_i| \in \sigma, \tag{38}$$

$$\text{(L2)} \quad \bar{\sigma} = \sigma, \tag{39}$$

$$\text{(L3)} \quad s_1(\sigma) = \lambda_1 + \lambda_2 + \lambda_3 \geqslant 0, \tag{40}$$

$$\text{(L4)} \quad [s_1(\sigma)]^2 \leqslant 3s_2(\sigma), \tag{41}$$

are necessary and sufficient for σ to be the spectrum of a nonnegative matrix of order 3.

Corollary 9. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ be a family of complex numbers such that $\sigma = \bar{\sigma}$ and let $P(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = x^3 + k_1x^2 + k_2x + k_3$. Then, the sets of conditions* (L1), (L3) *and* (L4) (*see* (38), (40), (41)) *and* (a)–(c) (*see* (34)–(36)) *are equivalent.*

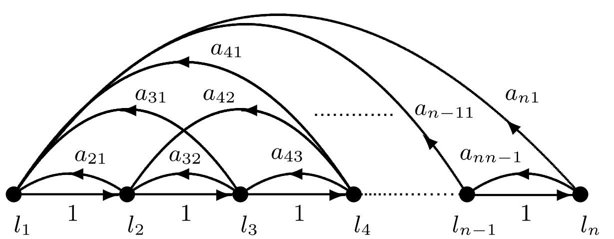
5. EBL digraphs and matrices

The matrices given in the above sections to obtain particular realizations share the characteristic of being nonnegative lower Hessenberg matrices with ones in the supradiagonal. These matrices and their associated weighted digraphs are an important tool which, together with the Coefficient Theorem, allows us to obtain results about realizability.

Definition 10. Let $G = (V, E, w)$ be a digraph with $V = \{v_1, \ldots, v_n\}$. We shall say that G is an EBL digraph (from the Spanish Estructura Básica Lineal, i.e., lineal basic structure) if $(v_i, v_j) \notin E$, for $j > i + 1$, and $(v_i, v_{i+1}) \in E$ with $w(v_i, v_{i+1}) = 1$, for $i = 1, \ldots, n - 1$.

We shall say that a matrix is EBL if it is the adjacency matrix of an EBL digraph. We shall say that a polynomial has an EBL realization or is EBL realizable when it is the characteristic polynomial of an EBL matrix, or equivalently, of an EBL digraph.

EBL digraphs have a notoriously simplified structure. They are made up of a fixed path $p = v_1v_2 \cdots v_n$ consecutively covering all the vertices of the digraph with arcs of weight 1. The only possible cycles $v_{i+1}v_{i+2} \cdots v_{i+r}v_{i+1}$ are built on the path p covering r consecutive vertices and an arc (v_{i+r}, v_{i+1}) closing the cycle. The weight $c_{i+1,\ldots,i+r}$ of these cycles is equal to the weight of the closing arc which we denote by $a_{i+r,i+1}$. The figure below is a graphic representation of such EBL digraphs



where the weights of the arcs are indicated. The possible loops are not shown but their weights l_i are associated with the corresponding vertices. According to this notation the EBL adjacency matrix of an EBL digraph is

$$\begin{pmatrix} l_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & l_2 & 1 & 0 & \cdots & 0 & 0 \\ a_{31} & a_{32} & l_3 & 1 & \ddots & \vdots & \vdots \\ a_{41} & a_{42} & a_{43} & l_4 & \ddots & 0 & \vdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ a_{n-11} & \ddots & \ddots & \ddots & \ddots & l_{n-1} & 1 \\ a_{n1} & \cdots & \cdots & \cdots & \cdots & a_{nn-1} & l_n \end{pmatrix}$$

$$= \begin{pmatrix} l_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ c_{12} & l_2 & 1 & 0 & \cdots & 0 & 0 \\ c_{123} & c_{23} & l_3 & 1 & \ddots & \vdots & \vdots \\ c_{1234} & c_{234} & c_{34} & l_4 & \ddots & 0 & \vdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ c_{1\ldots n-1} & \ddots & \ddots & \ddots & \ddots & l_{n-1} & 1 \\ c_{1\ldots n} & c_{2\ldots n} & c_{3\ldots n} & \cdots & \cdots & c_{n-1n} & l_n \end{pmatrix}. \tag{42}$$

Note that the weights of the existing 2-cycles in the EBL digraph occupy the first subdiagonal of the EBL matrix, the weights of the 3-cycles the second subdiagonal, etc.

We shall now look at some results about realizability in whose proofs EBL digraphs are used.

Theorem 11. *Let $P(x) = x^n + k_1x^{n-1} + \cdots + k_n$ be a polynomial with an EBL realization. Then the polynomial $x^n + \tilde{k}_1x^{n-1} + \cdots + \tilde{k}_n$ with $\tilde{k}_i \leqslant k_i$, for $i = 1, \ldots, n$, also has an EBL realization.*

Proof. Let A be an EBL matricial realization of $P(x)$

$$A = \begin{pmatrix} a_{11} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix} \tag{43}$$

and let G be the associated digraph with vertices $\{v_1, \ldots, v_n\}$.

We shall prove, by induction over n, that given $\epsilon_1 \geqslant 0$ and an integer i with $1 \leqslant i \leqslant n$, the polynomial $P(x) - \epsilon_1 x^{n-i}$ is EBL realizable by a matrix of the form

$$\begin{pmatrix} a_{11} & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ a_{i1}+\epsilon_1 & a_{i2} & \ddots & \ddots & \ddots & \vdots \\ a_{i+1,1}+\epsilon_2 & a_{i+1,2} & & \ddots & \ddots & 0 \\ \vdots & \vdots & & & \ddots & 1 \\ a_{n1}+\epsilon_{n-i+1} & a_{n2} & \cdots & \cdots & \cdots & a_{nn} \end{pmatrix} \tag{44}$$

where we have just modified the entries $(r, 1)$ of A, for $r = i, \ldots, n$, by adding $\epsilon_j \geqslant 0$, $j = 1, \ldots, n-i+1$.

The result is clear for $n = 1$. In the general case, we shall reach a linear system of equations in the unknowns ϵ_j, for $j = 2, \ldots, n-i+1$, whose solution is nonnegative.

Let A_j be the square submatrix of A formed by its last $n-j+1$ rows and columns and let $C_{i,j}$ be the coefficient of $x^{n-j+1-i}$ of the characteristic polynomial of A_j. The Coefficient Theorem and the particular structure of the digraph G allow us to obtain the equality

$$\begin{aligned} C_{i,j} = {} & -a_{jj}C_{i-1,j+1} - a_{j+1,j}C_{i-2,j+2} - \cdots \\ & -a_{j+i-2,j}C_{1,j+i-1} - a_{j+i-1,j} + C_{i,j+1}. \end{aligned} \tag{45}$$

Note that the indices i and j must verify $1 \leqslant i, j \leqslant n$ and $i + j \leqslant n$, because the last summand $C_{i,j+1}$ of (45) includes the contribution of the i-cycle $v_{j+1}v_{j+2}\cdots v_{j+i}v_{j+1}$, which requires the existence of the vertex v_{j+i}. We extend this equality for $C_{i,n-i+1}$ by putting

$$C_{i,n-i+2} = 0. \tag{46}$$

Let us return to the matrix (44). As the incorporation of ϵ_1 reduces the coefficient k_i of x^{n-i} by ϵ_1 but increases the coefficient k_{i+1} of x^{n-i-1} by $\epsilon_1(a_{i+1,i+1} + \cdots + a_{nn})$, then ϵ_2 must be

$$\epsilon_2 = \epsilon_1(a_{i+1,i+1} + \cdots + a_{nn}) = -\epsilon_1 C_{1,i+1} \geqslant 0. \tag{47}$$

Similarly, having fixed ϵ_1 and ϵ_2, the fitting of the coefficient of x^{n-i-2} to the value of k_{i+2} means that

$$\epsilon_3 = -\epsilon_2 C_{1,i+2} - \epsilon_1 C_{2,i+1}. \tag{48}$$

In general, we have

$$\epsilon_{j+1} = -\epsilon_j C_{1,j+i} - \epsilon_{j-1} C_{2,j+i-1} - \cdots - \epsilon_2 C_{j-1,i+2} - \epsilon_1 C_{j,i+1}, \quad 1 \leqslant j \leqslant n-i. \tag{49}$$

To show that $\epsilon_{j+1} \geqslant 0$, $1 \leqslant j \leqslant n-i$, the first j equations of (49) are matricially written as follows

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ C_{1,i+2} & 1 & \ddots & & \vdots \\ C_{2,i+2} & C_{1,i+3} & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ C_{j-1,i+2} & C_{j-2,i+3} & \cdots & C_{1,i+j} & 1 \end{pmatrix} \begin{pmatrix} \epsilon_2 \\ \epsilon_3 \\ \vdots \\ \vdots \\ \epsilon_{j+1} \end{pmatrix} = -\epsilon_1 \begin{pmatrix} C_{1,i+1} \\ C_{2,i+1} \\ \vdots \\ \vdots \\ C_{j,i+1} \end{pmatrix}. \tag{50}$$

Cramer's rule assures that

$$\epsilon_{j+1} = -\epsilon_1 \det \begin{pmatrix} 1 & 0 & \cdots & 0 & C_{1,i+1} \\ C_{1,i+2} & 1 & \ddots & \vdots & C_{2,i+1} \\ C_{2,i+2} & C_{1,i+3} & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & \vdots \\ C_{j-1,i+2} & C_{j-2,i+3} & \cdots & C_{1,i+j} & C_{j,i+1} \end{pmatrix}. \tag{51}$$

Applying (45) on the last column of the matrix from (51) and using elementary properties of the determinants we obtain

$$\epsilon_{j+1} = -\epsilon_1 \det \begin{pmatrix} 1 & 0 & \cdots & 0 & C_{1,i+2} \\ C_{1,i+2} & 1 & \ddots & \vdots & C_{2,i+2} \\ C_{2,i+2} & C_{1,i+3} & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & \vdots \\ C_{j-1,i+2} & C_{j-2,i+3} & \cdots & C_{1,i+j} & C_{j,i+2} \end{pmatrix} + \epsilon_1 a_{i+j,i+1}. \tag{52}$$

Observe that we have a nonnegative "residual term" $\epsilon_1 a_{i+j,i+1}$. In what follows we shall group the nonnegative values of the different nonnegative residual terms that we have and we shall denote them by "NRT". Note also that if $i + j = n$, then the entry (j, j) of the matrix from (52) is $C_{j,i+2} = 0$ because of (46).

To abbreviate the expressions that will appear, for a vector $\mathbf{v} \in \mathbb{R}^j$, we denote by $S_k(\mathbf{v})$ to the matrix of size $j \times j$ where

$$(0, \ldots, 0, 1, C_{1,i+p+2}, C_{2,i+p+2}, \ldots, C_{j-p,i+p+2})^{\mathrm{T}} \tag{53}$$

is the p-column, for $1 \leqslant p \leqslant k-1$, $\mathbf{v}$ is the k-column,

$$(0, \ldots, 0, 1, C_{1,i+q+1}, C_{2,i+q+1}, \ldots, C_{j-q,i+q+1})^{\mathrm{T}} \tag{54}$$

is the q-column, for $k+1 \leqslant q \leqslant j-1$, and

$$(C_{1,i+2}, C_{2,i+2}, \ldots, C_{j-1,i+2}, C_{j,i+2})^{\mathrm{T}} \tag{55}$$

is the j-column. Now we rewrite the columns, successively from the first to the penultimate, with a similar process to the one realized from (51) to (52). Realizing this process to the columns $1, \ldots, k-1$ we obtain

$$\epsilon_{j+1} = -\epsilon_1 \det S_k((0, \ldots, 1, C_{1,i+k+1}, C_{2,i+k+1}, \ldots, C_{j-k,i+k+1})^{\mathrm{T}}) + NRT. \tag{56}$$

Now, applying the equalities (45) to the entries of the k-column on the matrix from (56), we have

$$\begin{aligned} \epsilon_{j+1} = &-\epsilon_1 \det S_k((0, \ldots, 1, C_{1,i+k+2}, C_{2,i+k+2}, \ldots, C_{j-k,i+k+2})^{\mathrm{T}}) \\ &+ \epsilon_1 \det S_k((0, \ldots, 0, a_{i+j,i+k+1})^{\mathrm{T}}) + NRT. \end{aligned} \tag{57}$$

Note that changing over the k-column and the j-column we get

$$\epsilon_1 \det S_k((0,\ldots,0,a_{i+j,i+k+1})^{\mathrm{T}}) = -\epsilon_1 a_{i+j,i+k+1} \det \begin{pmatrix} 1 & 0 & \cdots & 0 & C_{1,i+2} \\ C_{1,i+3} & 1 & \ddots & 0 & C_{2,i+2} \\ C_{2,i+3} & C_{1,i+4} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & C_{k-1,i+2} \\ C_{k-1,i+3} & C_{k-2,i+4} & \cdots & C_{1,i+k+1} & C_{k,i+2} \end{pmatrix} \tag{58}$$

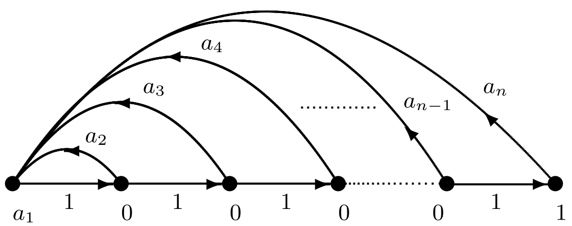
which is nonnegative by analogy with (51) and by the induction hypothesis. Hence it is included in NRT and we can write

$$\epsilon_{j+1} = -\epsilon_1 \det \begin{pmatrix} 1 & 0 & 0 & \cdots & C_{1,i+2} \\ C_{1,i+3} & 1 & 0 & \ddots & C_{2,i+2} \\ C_{2,i+3} & C_{1,i+4} & \ddots & \ddots & C_{3,i+2} \\ \vdots & \vdots & & 1 & \vdots \\ C_{j-1,i+3} & C_{j-2,i+4} & \cdots & C_{1,i+j+1} & C_{j,i+2} \end{pmatrix} + NRT. \tag{59}$$

With respect to (51), we have increased by one the indices of the columns and we have also added the summand NRT. Then, repeating the whole process we get to a point where $i+j+1>n$ and the last row of the corresponding matrix from (59) is zero, thus only NRT remains. $\square$

Theorem 12. *Given $k_2, k_3, \ldots, k_n$ real numbers, then there exists a real number k_1 such that $P(x) = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_n$ is a realizable polynomial.*

Proof. We shall see how, taking $-k_1$ sufficiently large, it is possible to find an EBL realization of $P(x)$ of the type



$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ a_2 & 0 & 1 & \cdots & 0 & 0 \\ a_3 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 1 \\ a_n & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{60}$$

In this EBL digraph the only cyclic structures with r vertices are $CS_{1(r-1)}$ and CS_r. Now a_i denotes the weight of the only i-cycle (which connects the vertices $v_1, v_2, \ldots, v_i$). Identifying the coefficients of $P(x)$ with those of the characteristic polynomial of the matrix given in (60) we have

$$\begin{aligned} k_r &= a_{r-1} - a_r, \quad r = 2,3,\ldots,n, \\ k_1 &= -a_1 - 1. \end{aligned} \tag{61}$$

This linear system can be rewritten as

$$a_r = -1 - k_1 - \cdots - k_r, \quad r = 1, \ldots, n. \tag{62}$$

The result follows taking $-k_1$ sufficiently large. □

5.1. *EBL realizations for n = 3*

In Section 4, we proved that any realizable polynomial of degree 2 or 3 has an EBL realization. Let us see that for a realizable polynomial of degree 3, it is possible to find an EBL realization by modifying a known realization.

Without loss of generality, a nonnegative matricial realization of the polynomial $x^3 + k_1x^2 + k_2x + k_3$

$$\begin{pmatrix} l_1 & a_{12} & a_{13} \\ a_{21} & l_2 & a_{23} \\ a_{31} & a_{32} & l_3 \end{pmatrix} \tag{63}$$

with increasing diagonal $l_1 \leqslant l_2 \leqslant l_3$ can be used as a starting point. We will now build an EBL realization of $P(x)$ with the same loops, so that k_1 is not modified. According to the Coefficient Theorem

$$\begin{aligned} k_2 &= f_2(l_1, l_2, l_3) - \sum_{CS_2} c_{ij} \quad \text{where } c_{ij} = a_{ij}a_{ji} \\ k_3 &= -f_3(l_1, l_2, l_3) + \sum_{CS_{12}} l_i c_{jq} - \sum_{CS_3} c_{ijq} \quad \text{where } c_{ijr} = a_{ij}a_{jr}a_{ri}. \end{aligned} \tag{64}$$

In order to preserve k_2, the sum of the weights of the 2-cycles must be preserved. As for k_3, the key is in the summand referred to CS_{12}. To avoid loss of positivity of this summand, we focus the weights of the 2-cycles at the entry $(2, 1)$ of the EBL matrix, and therefore opposite the biggest loop, thus maximizing the contribution of this summand to k_3. To adjust k_3 it is enough to add the necessary weight of the 3-cycle. The EBL matrix obtained is then

$$\begin{pmatrix} l_1 & 1 & 0 \\ d & l_2 & 1 \\ t & 0 & l_3 \end{pmatrix} \quad \text{where} \begin{cases} d = c_{12} + c_{13} + c_{23}, \\ t = c_{123} + c_{132} + c_{13}(l_3 - l_2) + c_{23}(l_3 - l_1). \end{cases} \tag{65}$$

Similar ideas to these can be used for $n = 4$, as we shall see below.

5.2. *EBL realizations for n = 4*

Proceeding as in the case $n = 3$, given a realizable polynomial $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$ and a nonnegative matricial realization

$$\begin{pmatrix} l_1 & a_{12} & a_{13} & a_{14} \\ a_{21} & l_2 & a_{23} & a_{24} \\ a_{31} & a_{32} & l_3 & a_{34} \\ a_{41} & a_{42} & a_{43} & l_4 \end{pmatrix}, \tag{66}$$

with $l_1 \leqslant l_2 \leqslant l_3 \leqslant l_4$, we shall find an EBL realization of $P(x)$. In accordance with the Coefficient Theorem

$$\begin{aligned} k_2 &= f_2(l_1, l_2, l_3, l_4) - \sum_{CS_2} c_{ij}, \\ k_3 &= -f_3(l_1, l_2, l_3, l_4) + \sum_{CS_{12}} l_i c_{jq} - \sum_{CS_3} c_{ijq}, \\ k_4 &= f_4(l_1, l_2, l_3, l_4) - \sum_{CS_{112}} l_i l_j c_{qr} + \sum_{CS_{13}} l_i c_{jqr} + \sum_{CS_{22}} c_{ij} c_{qr} - \sum_{CS_4} c_{ijqr}. \end{aligned} \tag{67}$$

We shall proceed with the following criteria:

1. Preserve the loop weights and the sum of the 2-cycle weights of the original realization.
2. Concentrate the 2-cycle weights in such a way that there is no loss of positivity in the summand referred to CS_{12}.
3. Concentrate the 2-cycle weights in such a way that the possible loss of positivity in the summand referred to CS_{22} is compensated for by the disappearance of 4-cycles in the summand referred to CS_4.

In order to see how to apply the third of these criteria, let us suppose that the 2-cycle weights c_{ij}, c_{jq}, c_{qr} and c_{ri} (Fig. 1) are all positive in the initial realization. Thus the summands corresponding to CS_{22} and to CS_4 will also be positive. If any of these 2-cycles are suppressed in the concentration of the 2-cycle weights, a loss of negativity occurs in the summands referred to CS_4 because of the disappearance of the 4-cycles (see Figs. 2 and 3).

The following lemma gives a lower bound for this loss of negativity.

Lemma 13. *Using the previous notation*

$$c_{ijqr} + c_{irqj} \geqslant 2\sqrt{c_{ij} c_{jq} c_{qr} c_{ri}} \geqslant 2 \min\{c_{ij} c_{qr}, c_{ir} c_{qj}\}. \tag{68}$$

Proof. Given that

$$c_{irqj} = a_{ji} a_{qj} a_{rq} a_{ir} = \frac{c_{ij} c_{jq} c_{qr} c_{ri}}{a_{ij} a_{jq} a_{qr} a_{ri}} = \frac{c_{ij} c_{jq} c_{qr} c_{ri}}{c_{ijqr}} \tag{69}$$

it is enough to bear in mind that the map $x + c/x$ with $c > 0$ attained its minimum in $(0, +\infty)$ at the point $x = \sqrt{c}$ and that this minimum value is $2\sqrt{c}$. □

This means that, on moving the 2-cycle weights, the "large" pair of 2-cycles should be kept opposite each other so that loss of positivity produced by the cancelling of some 2-cycles will be compensated for by the loss of negativity in the corresponding 4-cycles.

The pattern of behaviour just described in order to respect the third of the criteria set out at the beginning of the section, leads us to give the EBL realizations separately in three cases,

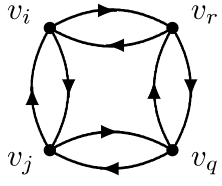


Fig. 1. $c_{ij}, c_{jq}, c_{qr}, c_{ri}$.

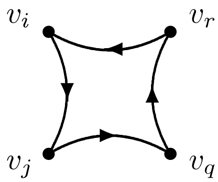


Fig. 2. c_{ijqr}.

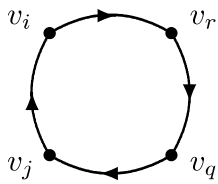


Fig. 3. c_{irqj}.

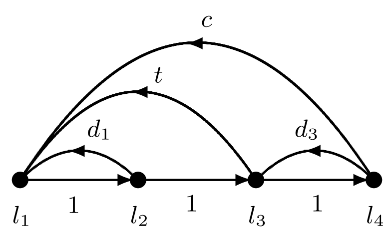
depending on what the "large" pair in the CS_{22} is, as can be seen in the proof of the following result.

Theorem 14. *Every realizable polynomial of degree* 4 *is EBL realizable*.

Proof. Let $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$ be a polynomial with matricial realization as (66) with increasing diagonal, $l_1 \leqslant l_2 \leqslant l_3 \leqslant l_4$. We consider the following cases:

First case: $c_{12}c_{34} = \max\{c_{ij}c_{qr}\}$. An EBL realization is

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ t & 0 & l_3 & 1 \\ c & 0 & d_3 & l_4 \end{pmatrix}$$



where

$$\begin{aligned}
d_1 &= c_{12} + c_{13} + c_{14} + c_{23} + c_{24}, \\
d_3 &= c_{34}, \\
t &= \sum_{CS_3} c_{ijq} + \underbrace{c_{13}(l_3 - l_2)}_{t_{13}} + \underbrace{c_{14}(l_4 - l_2)}_{t_{14}} + \underbrace{c_{23}(l_3 - l_1)}_{t_{23}} + \underbrace{c_{24}(l_4 - l_1)}_{t_{24}}, \qquad (70) \\
c &= \underbrace{d_1d_3 - \sum_{CS_{22}} c_{ij}c_{qr} + \sum_{CS_4} c_{ijqr}}_{[e.1]} \\
&\quad \underbrace{+ tl_4 - d_1l_3l_4 - d_3l_1l_2 - \left(\sum_{CS_{13}} l_i c_{jqr} - \sum_{CS_{112}} l_i l_j c_{qr} \right)}_{[e.2]}.
\end{aligned}$$

It is clear that d_1, d_3 and t are nonnegative. The expression of t has been built by adding the necessary weights to the 3-cycles of the original realization, in order to preserve the coefficient k_3 of the polynomial $P(x)$. Each braced summand describes the necessary weight for compensating the increase in k_3 caused by the change in position of the corresponding 2-cycles. For instance, t_{14} is the weight that compensates for the increase in k_3 due to the displacement of c_{14} from the vertices v_1 and v_4 in the original realization to the vertices v_1 and v_2 in the EBL realization.

The expression of c represents the difference between the part of k_4 generated by d_1, d_3 and t and the k_4 of the given polynomial. The expression $[e.1]$ is nonnegative because:

(1) $c_{12}c_{34} < d_1 d_3$ by definition of d_1 and d_3, and
(2) $c_{23}c_{14} \leqslant c_{1234} + c_{1432}$ and $c_{13}c_{24} \leqslant c_{1243} + c_{1342}$ by the above lemma and because, in this case, $c_{12}c_{34}$ is the largest term in CS_{22}.

Let us now see that $[e.2]$ is also nonnegative. Taking into account that

$$\begin{aligned} tl_4 = & \sum_{CS_{13}} l_i c_{jqr} + \sum_{CS_{13}} (l_4 - l_i) c_{jqr} + (t_{14} + t_{24})(l_4 - l_3) \\ & + t_{13} l_4 + t_{14} l_3 + t_{23} l_4 + t_{24} l_3 \end{aligned} \tag{71}$$

and that

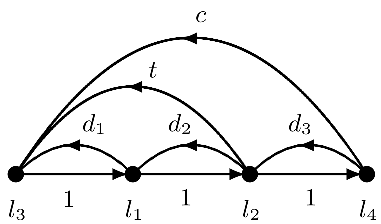
$$-\sum_{CS_{112}} l_i l_j c_{qr} = -d_1 l_3 l_4 - d_3 l_1 l_2 + t_{13} l_4 + t_{14} l_3 + t_{23} l_4 + t_{24} l_3 \tag{72}$$

$[e.2]$ can be expressed as follows

$$[e.2] = \sum_{CS_{13}} (l_4 - l_i) c_{jqr} + (t_{14} + t_{24})(l_4 - l_3) \geqslant 0. \tag{73}$$

Second case: $c_{13}c_{24} = \max\{c_{ij}c_{qr}\}$. An EBL realization is

$$\begin{pmatrix} l_3 & 1 & 0 & 0 \\ d_1 & l_1 & 1 & 0 \\ t & d_2 & l_2 & 1 \\ c & 0 & d_3 & l_4 \end{pmatrix}$$

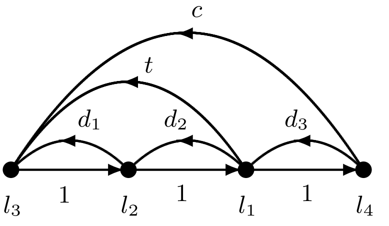


where

$$\begin{aligned} d_1 &= c_{13} + c_{14} + c_{23} + c_{34} \\ d_2 &= c_{12} \\ d_3 &= c_{24} \\ t &= \sum_{CS_3} c_{ijq} + \underbrace{c_{14}(l_4 - l_3)}_{t_{14}} + \underbrace{c_{23}(l_2 - l_1)}_{t_{23}} + \underbrace{c_{34}(l_4 - l_1)}_{t_{34}} \\ c &= \underbrace{d_1 d_3 - \sum_{CS_{22}} c_{ij} c_{qr} + \sum_{CS_4} c_{ijqr}}_{[e.1]} + \underbrace{\sum_{CS_{13}} (l_4 - l_i) c_{jqr} + (t_{14} + t_{34})(l_4 - l_2)}_{[e.2]}. \end{aligned} \tag{74}$$

Third case: $c_{14}c_{23} = \max\{c_{ij}c_{qr}\}$. An EBL realization is:

$$\begin{pmatrix} l_3 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ t & d_2 & l_1 & 1 \\ c & 0 & d_3 & l_4 \end{pmatrix}$$



where

$$\begin{aligned} d_1 &= c_{23} + c_{24} + c_{34} \\ d_2 &= c_{12} + c_{13} \\ d_3 &= c_{14} \\ t &= \sum_{CS_3} c_{ijq} + \underbrace{c_{13}(l_3 - l_2)}_{t_{13}} + \underbrace{c_{24}(l_4 - l_3)}_{t_{24}} + \underbrace{c_{34}(l_4 - l_2)}_{t_{34}} \\ c &= \underbrace{d_1 d_3 - \sum_{CS_{22}} c_{ij}c_{qr} + \sum_{CS_4} c_{ijqr}}_{[e.1]} \\ &\quad + \underbrace{\sum_{CS_{13}} (l_4 - l_i)c_{jqr} + (t_{24} + t_{34})(l_4 - l_1)}_{[e.2]}. \quad \square \end{aligned} \tag{75}$$

Remark 15. Because every realizable polynomial of degree 4 admits an EBL realization with the entry $(4, 2)$ zero, in what follows we shall only consider EBL realizations with this feature.

Remark 16. The 4-cycles only affect the independent term of the characteristic polynomial. So, if we are interested in obtaining the maximum k_4, given k_1, k_2 and k_3, then we can assume EBL realizations with the entry $(4, 1)$ zero.

6. The case $n = 4$

Given a polynomial, $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$, Theorem 3 gives necessary conditions over three of its coefficients for $P(x)$ to be realizable:

(a) $k_1 \leqslant 0$;

(b) $k_2 \leqslant k_2^{\max}(k_1) = \frac{3}{8}k_1^2$; (76)

(c) $k_3 \leqslant k_3^{\max}(k_1, k_2) = \begin{cases} \frac{k_1k_2}{2} + \frac{1}{8}\left(\left(k_1^2 - \frac{8k_2}{3}\right)^{3/2} - k_1^3\right) & \text{if } k_2 > 0, \\ k_1k_2 - \frac{k_1^3}{4} & \text{if } k_2 \leqslant 0. \end{cases}$

Theorem 3 also says that given k_1, k_2 and k_3 verifying (76) we can find a realizable polynomial of the form $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$. This means that the shape and the position, except vertical translations, of the graph of $P(x)$ are known. Hence, in order to characterize the polynomials of degree 4 realizable we need to describe $k_4^{\max}(k_1, k_2, k_3)$. Firstly, let us see that $k_4^{\max}(k_1, k_2, k_3)$ exists.

Theorem 17. *Let k_1, k_2 and k_3 verify the necessary conditions* (76). *Then there exists a realizable polynomial $x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$ with $k_4 = k_4^{\max}(k_1, k_2, k_3)$.*

Proof. Let us see that there exists an EBL matrix, see Theorem 14 and Remarks 15 and 16,

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ t & d_2 & l_3 & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix} \tag{77}$$

whose characteristic polynomial has the desired k_4. From the Coefficient Theorem we know

$$\begin{aligned} k_1 &= -\sum_{i=1}^{4} l_1 \Rightarrow l_i \leqslant -k_1, \quad i = 1, 2, 3, 4 \\ k_2 &= f_2(l_1, l_2, l_3, l_4) - \sum_{i=1}^{3} d_i \Rightarrow d_i \leqslant \frac{3}{8}k_1^2 - k_2, \quad i = 1, 2, 3 \\ k_3 &= -f_3(l_1, l_2, l_3, l_4) + d_1(l_3 + l_4) + d_2(l_1 + l_4) + d_3(l_1 + l_2) - t \\ &\Rightarrow t \leqslant -\frac{k_1^3}{16} - 2k_1\left(\frac{3}{8}k_1^2 - k_2\right) + |k_3| \end{aligned} \tag{78}$$

which means that given k_1, k_2 and k_3 the entries of the above EBL matrix are bounded. This assures the result because the determinant, k_4, is a continuous function of the entries of the matrix. □

Given a realizable polynomial $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$, its inflexion points have x-coordinates, that we denote by x_{li} (*left inflexion*) and x_{ri} (*right inflexion*), with values

$$x_{li}(k_1, k_2) = -\frac{k_1}{4} - \frac{1}{\sqrt{6}}\sqrt{\frac{3}{8}k_1^2 - k_2} \quad \text{and} \quad x_{ri}(k_1, k_2) = -\frac{k_1}{4} + \frac{1}{\sqrt{6}}\sqrt{\frac{3}{8}k_1^2 - k_2}. \tag{79}$$

We call the real number $-k_1/4$ *the centre of the polynomial* $P(x)$, that is, the midpoint of the segment that joins the x-coordinates of the inflexion points of the polynomial. The Taylor expansion of $P(x)$ at its centre

$$P(x) = \left(x + \frac{k_1}{4}\right)^4 - \left(\frac{3}{8}k_1^2 - k_2\right)\left(x + \frac{k_1}{4}\right)^2 + P'\left(-\frac{k_1}{4}\right)\left(x + \frac{k_1}{4}\right) + P\left(-\frac{k_1}{4}\right) \tag{80}$$

shows that the position of the graph of $P(x)$ is determined by $-k_1/4$ and its shape by the values

$$k_2^{\max}(k_1) - k_2 = \frac{3}{8}k_1^2 - k_2 \tag{81}$$

and

$$P'\left(-\frac{k_1}{4}\right) = k_3 + \frac{k_1}{2}\left(\frac{k_1^2}{4} - k_2\right). \tag{82}$$

The inequalities given in (76) have the following graphical implications:

- The condition $k_1 \leqslant 0$ means that the centre of the polynomial is in $[0, +\infty)$.
- The condition $k_2 \leqslant k_2^{\max}(k_1)$ means that $P(x)$ has two inflexion points (equal when $k_2 = k_2^{\max}(k_1)$) and that the distance between their x-coordinates depends on (81).
- The condition $k_3 \leqslant k_3^{\max}(k_1, k_2)$ means that the slope of the tangent at the centre is smaller than a bound. When this tangent is horizontal the graph of $P(x)$ is symmetric with respect to the line $x = -k_1/4$ and we shall say that $P(x)$ is balanced.

Definition 18. We say that a realizable polynomial $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$ is balanced or in equilibrium when $P'(-k_1/4) = 0$ and we denote by $k_3^{\text{eq}}(k_1, k_2)$ the value of the coefficient of x of this polynomial, that is,

$$k_3^{\text{eq}}(k_1, k_2) = -\frac{k_1}{2}\left(\frac{k_1^2}{4} - k_2\right). \tag{83}$$

The expression (83) allows us to rewrite the $k_3^{\max}(k_1, k_2)$ given in (76) as

$$k_3^{\max}(k_1, k_2) = \begin{cases} k_3^{\text{eq}}(k_1, k_2) + \left(\frac{2}{3}(k_2^{\max}(k_1) - k_2)\right)^{3/2} & \text{if } k_2 > 0, \\ 2k_3^{\text{eq}}(k_1, k_2) & \text{if } k_2 \leqslant 0. \end{cases} \tag{84}$$

Note that $\left(\frac{2}{3}(k_2^{\max}(k_1) - k_2)\right)^{1/2} = x_{ri}(k_1, k_2) - x_{li}(k_1, k_2)$.

Our objective is to obtain the value $k_4^{\max}(k_1, k_2, k_3)$ and it will be given as the determinant of a particular matricial realization.

In what follows, when there is no doubt from the context, we will omit the dependency of functions such as $k_2^{\max}$, k_3^{eq} or x_{li} from the coefficients.

Let us see some restrictions about the choice of the diagonal elements of nonnegative realizations of polynomials of degree 4 and some results related to these restrictions.

The necessary conditions given in (76) say that the loop weights l_i are bounded by $-k_1$. When $k_2 > 0$ we have more restrictions on the choice of these weights because the positivity of k_2 only comes from CS_{11}. Without being precise, for k_2 close to $k_2^{\max}(k_1)$ the loop weights will be close to being equally distributed. The next result specifies these ideas.

Lemma 19. *Let $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$ be a realizable polynomial with $k_2 > 0$. Then, with the introduced notations, the loop weights l_i of any realization of $P(x)$ must verify*

$$\max_{1\leqslant i\leqslant 4}\{l_i\} \leqslant 2x_{ri} - x_{li}. \tag{85}$$

Proof. According to the Coefficient Theorem

$$k_2 = f_2(l_1, l_2, l_3, l_4) - \sum_{CS_2} c_{ij}. \tag{86}$$

Let us assume $l_4 = \max_{1\leqslant i\leqslant 4}\{l_i\}$. Note that $l_4 \geqslant -k_1/4$ and that $f_2(l_1, l_2, l_3)$ attains its maximum when $l_1 = l_2 = l_3$. Then we have

$$f_2(l_1, l_2, l_3, l_4) = l_4(l_1 + l_2 + l_3) + f_2(l_1, l_2, l_3) \leqslant l_4(-k_1 - l_4) + \frac{1}{3}(-k_1 - l_4)^2. \quad (87)$$

The result follows by solving the equation $k_2 = l_4(-k_1 - l_4) + \frac{1}{3}(-k_1 - l_4)^2$ on l_4:

$$l_4 = -\frac{k_1}{4} + \frac{3}{\sqrt{6}}\sqrt{\frac{3}{8}k_1^2 - k_2} = 2x_{ri} - x_{li}. \qquad \square \quad (88)$$

In what follows we denote the largest loop weight of any realization with fixed k_1 and k_2 by $l^{\max}(k_1, k_2)$, so from the above result

$$l^{\max}(k_1, k_2) = \begin{cases} -\frac{k_1}{4} + \frac{3}{\sqrt{6}}\sqrt{\frac{3}{8}k_1^2 - k_2} & \text{if } k_2 > 0, \\ -k_1 & \text{if } k_2 \leqslant 0. \end{cases} \quad (89)$$

The value $l^{\max}(k_1, k_2)$ will be significant in the study of $k_4^{\max}(k_1, k_2, k_3)$.

Lemma 20. *Let $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$ be a realizable polynomial with two non real complex roots.*

(1) *If r is a double real root of P, then any matricial realization of P, EBL or not, has r as a diagonal element, and therefore*

$$r \leqslant l^{\max}(k_1, k_2). \quad (90)$$

(2) *If the two real roots of P are larger than $l^{\max}(k_1, k_2)$, then P only admits irreducible realizations.*

Proof. The existence of two non real complex roots implies that the Frobenius normal form of any matricial realization must have an irreducible component of size greater than or equal to 3.

(1) By the Frobenius Theorem, P can only have reducible realizations with irreducible components of sizes 3 and 1. The component of size 1 is r and, therefore, r is a diagonal element of any realization of P.

(2) If P has a reducible realization, then P has an irreducible component of size 1 and none of its real roots is sufficiently small to be the diagonal element of this component. □

Theorem 21. *Let k_1, k_2 and k_3 verify the necessary conditions (76) and let $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{\max}(k_1, k_2, k_3)$. If $P(x)$ has an EBL realization with $l^{\max}(k_1, k_2)$ and $P(x) > 0$ for all $x < l^{\max}(k_1, k_2)$, then for every polynomial $\widetilde{P}(x) = x^4 + k_1x^3 + k_2x^2 + \tilde{k}_3x + k_4^{\max}(k_1, k_2, \tilde{k}_3)$ with $\tilde{k}_3 < k_3$ we have*

$$P(l^{\max}(k_1, k_2)) = \widetilde{P}(l^{\max}(k_1, k_2)). \quad (91)$$

Proof. Let

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ t & d_2 & l_3 & 1 \\ 0 & 0 & d_3 & l^{\max} \end{pmatrix} \quad (92)$$

be an EBL realization of $P(x)$. For each $\delta > 0$ the nonnegative matrix

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ t+\delta & d_2 & l_3 & 1 \\ 0 & 0 & d_3 & l^{\max} \end{pmatrix} \tag{93}$$

has characteristic polynomial

$$Q_\delta(x) = P(x) - \delta x + \delta l^{\max}. \tag{94}$$

These polynomials verify

$$Q_\delta(l^{\max}) = P(l^{\max}), \tag{95}$$
$$Q_\delta(x) > P(x) > 0 \quad \forall x < l^{\max}. \tag{96}$$

Let $\tilde{k}_3 < k_3$. Because

$$Q_{k_3-\tilde{k}_3}(x) = x^4 + k_1x^3 + k_2x^2 + \tilde{k}_3x + k_4^{\max}(k_1,k_2,k_3) + (k_3-\tilde{k}_3)l^{\max} \tag{97}$$

we have $Q_{k_3-\tilde{k}_3}(x) \leqslant \widetilde{P}(x)$ and $Q'_{k_3-\tilde{k}_3}(x) = \widetilde{P}'(x)$.

Let us assume that $Q_{k_3-\tilde{k}_3}(x) < \widetilde{P}(x)$, for all $x \in \mathbb{R}$. Let

$$\begin{pmatrix} \tilde{l}_1 & 1 & 0 & 0 \\ \tilde{d}_1 & \tilde{l}_2 & 1 & 0 \\ \tilde{t} & \tilde{d}_2 & \tilde{l}_3 & 1 \\ 0 & 0 & \tilde{d}_3 & \tilde{l}_4 \end{pmatrix} \tag{98}$$

be an EBL realization of $\widetilde{P}(x)$ and let

$$t_{\min} = \min\{k_3 - \tilde{k}_3, \tilde{t}\}. \tag{99}$$

Let $Q^*_{k_3-\tilde{k}_3}(x)$ and $\widetilde{P}^*(x)$ be the characteristic polynomials of the matrices

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ t+k_3-\tilde{k}_3-t_{\min} & d_2 & l_3 & 1 \\ 0 & 0 & d_3 & l^{\max} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{l}_1 & 1 & 0 & 0 \\ \tilde{d}_1 & \tilde{l}_2 & 1 & 0 \\ \tilde{t}-t_{\min} & \tilde{d}_2 & \tilde{l}_3 & 1 \\ 0 & 0 & \tilde{d}_3 & \tilde{l}_4 \end{pmatrix}, \tag{100}$$

respectively. Note that these matrices have been obtained from the EBL matrices of $Q_{k_3-\tilde{k}_3}(x)$ and $\widetilde{P}(x)$, respectively, when reducing the 3-cycle weight for $t_{\min}$. We have

$$Q^*_{k_3-\tilde{k}_3}(x) = Q_{k_3-\tilde{k}_3}(x) + t_{\min}x - t_{\min}l^{\max}, \tag{101}$$
$$\widetilde{P}^*(x) = \widetilde{P}(x) + t_{\min}x - t_{\min}\tilde{l}_4 \tag{102}$$

and

$$\widetilde{P}^*(x) - Q^*_{k_3-\tilde{k}_3}(x) = \widetilde{P}(x) - Q_{k_3-\tilde{k}_3}(x) + t_{\min}(l^{\max} - \tilde{l}_4) > 0. \tag{103}$$

If $t_{\min} = k_3 - \tilde{k}_3$, then $Q^*_{k_3-\tilde{k}_3} = P$ which contradicts the above inequality (103).

If $t_{\min} < k_3 - \tilde{k}_3$, then $t_{\min} = \tilde{t}$ and hence $\widetilde{P}^*$ admits the symmetric realization

$$\begin{pmatrix} \tilde{l}_1 & \sqrt{\tilde{d}_1} & 0 & 0 \\ \sqrt{\tilde{d}_1} & \tilde{l}_2 & \sqrt{\tilde{d}_2} & 0 \\ 0 & \sqrt{\tilde{d}_2} & \tilde{l}_3 & \sqrt{\tilde{d}_3} \\ 0 & 0 & \sqrt{\tilde{d}_3} & \tilde{l}_4 \end{pmatrix} \tag{104}$$

which guarantees that all the roots of $\widetilde{P}^*$ are real. Note that the weights of cyclic structure corresponding to realization (100) are equal to the weights of cyclic structure corresponding to realization (104).

Because $Q^*_{k_3-\tilde{k}_3}(x) = Q_{k_3-\tilde{k}_3-t_{\min}}(x)$, it follows from (103) and (96) that $\widetilde{P}^*(x) > 0$, for all $x \leqslant l^{\max}$. Therefore $\widetilde{P}^*(x)$ is positive on $(-\infty, -k_1/4]$ because $l^{\max} \geqslant -k_1/4$, and this goes against the real character of the roots of this polynomial. Hence the assumption that $Q_{k_3-\tilde{k}_3}(x) < \widetilde{P}(x)$ is false, see the line above (98). This combined with the assertion after (97) gives $Q_{k_3-\tilde{k}_3}(x) = \widetilde{P}(x)$, for all $x \in \mathbb{R}$. Finally, the result follows from (95). □

6.1. *The study of* $k_4^{\max}(k_1, k_2, k_3)$ *in some simple cases*

We will study the value $k_4^{\max}(k_1, k_2, k_3)$ when $k_3 = k_3^{\max}(k_1, k_2)$ or $k_2 = k_2^{\max}(k_1)$ or $k_1 = 0$.

In Section 3 it was seen that the constructed realizations, see (30), of the polynomials with $k_3 = k_3^{\max}(k_1, k_2)$ are strongly limited: there are no 3-cycles, the weight of all 2-cycles is focussed on 2-cycles connecting two vertices with loops of lowest weight and $l_3 = l_4 = l_4^{k_3^{\max}}(k_1, k_2)$.

These observations and the knowledge of the existence of an EBL realization for every realizable polynomial of degree 4 allow us to say that the polynomials corresponding to $k_3 = k_3^{\max}(k_1, k_2)$ can only have EBL realizations of one of the two types

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ 0 & 0 & l_4^{k_3^{\max}} & 1 \\ c & 0 & 0 & l_4^{k_3^{\max}} \end{pmatrix}, \quad \text{where}$$

$$\begin{cases} l_4^{k_3^{\max}} & = \begin{cases} x_{ri} & \text{if } k_2 > 0, \\ -\frac{k_1}{2} & \text{if } k_2 \leqslant 0, \end{cases} \\ l_1 + l_2 & = -k_1 - 2l_4^{k_3^{\max}} = \begin{cases} 2x_{li} & \text{if } k_2 > 0, \\ 0 & \text{if } k_2 \leqslant 0, \end{cases} \\ d_1 & = f_2\left(l_1, l_2, l_4^{k_3^{\max}}, l_4^{k_3^{\max}}\right) - k_2, \end{cases} \tag{105}$$

$$\begin{pmatrix} l_4^{k_3^{\max}} & 1 & 0 & 0 \\ d_1 & l_4^{k_3^{\max}} & 1 & 0 \\ 0 & d_2 & l_4^{k_3^{\max}} & 1 \\ c & 0 & d_3 & l_4^{k_3^{\max}} \end{pmatrix}, \quad \text{where}$$

$$\begin{cases} l_4^{k_3^{\max}} = -\frac{k_1}{4}, \\ d_1 + d_2 + d_3 = 6\left(l_4^{k_3^{\max}}\right)^2 - k_2 = k_2^{\max} - k_2, \end{cases} \tag{106}$$

with $c \geqslant 0$, in both cases. The second type is only possible when $k_2 = k_2^{\max}(k_1)$ or $k_1 = 0$. Note that every EBL realization corresponding to $k_4^{\max}$ must have $c = 0$.

First of all we study the case $k_1 < 0$ and $k_2 < k_2^{\max}(k_1)$, so we have realizations of the type (105). The EBL realization corresponding to $k_4^{\max}(k_1, k_2, k_3^{\max}(k_1, k_2))$ must have the weights of the loops verifying $l_1 = l_2$. Figs. 4 and 5 show the two possible general shapes of the graph of a realizable polynomial of degree 4 in this situation. The band between the inflexion points has been shaded.

For $k_2 > 0$, both inflexion points are in the semiplane $x > 0$ and the right local minimum has overlapped the right inflexion point. In this case, x_{ri} is the spectral radius and is a triple root of $P(x)$.

For $k_2 \leqslant 0$, the left inflexion point is in the semiplane $x \leqslant 0$ and the graph of $P(x)$ is characterized for being tangent to the x-axis at the local maximum (attained at $-k_1/2$) and for having ρ and $-\rho$ as roots, where

$$\rho = \sqrt{\frac{k_1^2}{4} - k_2} \tag{107}$$

is the spectral radius.

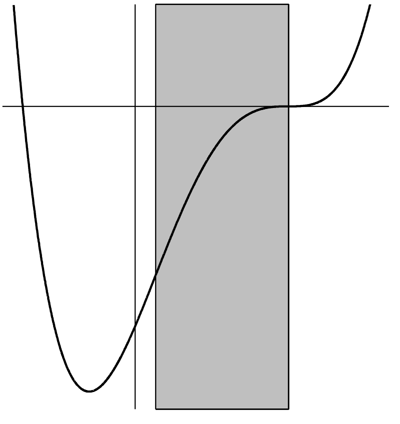


Fig. 4. $k_4^{\max}(k_1, k_2, k_3^{\max}), k_2 > 0$.

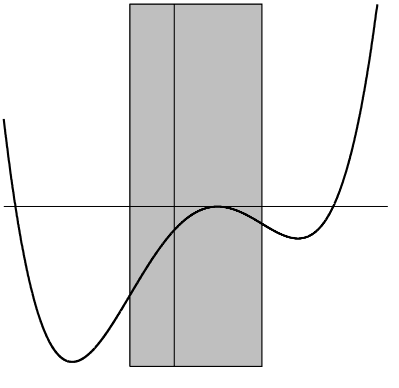


Fig. 5. $k_4^{\max}(k_1, k_2, k_3^{\max}), k_2 \leqslant 0$.

Finally we study simultaneously the cases $k_1 = 0$ and $k_2 = k_2^{\max}(k_1)$, so we have realizations of the type (106), because in this situation all the loop weights are equal. This means that all the loops have minimum weight. An EBL realization for $k_3^{\max}(k_1, k_2)$ can have the weights of the 2-cycles, d_1, d_2 and d_3, arbitrarily distributed. The EBL realizations for $k_4 = k_4^{\max}(k_1, k_2, k_3^{\max})$ are

$$\begin{pmatrix} l_4^{k_3^{\max}} & 1 & 0 & 0 \\ d_1 & l_4^{k_3^{\max}} & 1 & 0 \\ 0 & 0 & l_4^{k_3^{\max}} & 1 \\ 0 & 0 & d_1 & l_4^{k_3^{\max}} \end{pmatrix} \quad \text{where} \begin{cases} l_4^{k_3^{\max}} = -\frac{k_1}{4}, \\ d_1 = -\frac{k_1}{2}. \end{cases} \tag{108}$$

If we want to make k_3 smaller, then the corresponding EBL realization has the entry (3,1) non zero because the 2-cycle weights are determined by k_2 and any distribution of these weights keep us in the case $k_3^{\max}(k_1, k_2)$ because the loop weights are equally distributed. Hence, the only way of making k_3 smaller is by increasing the weight of the 3-cycles. The realization corresponding to $k_4 = k_4^{\max}(k_1, k_2, k_3)$ is

$$\begin{pmatrix} l_4^{k_3^{\max}} & 1 & 0 & 0 \\ d_1 & l_4^{k_3^{\max}} & 1 & 0 \\ t & 0 & l_4^{k_3^{\max}} & 1 \\ 0 & 0 & d_1 & l_4^{k_3^{\max}} \end{pmatrix} \quad \text{where} \begin{cases} l_4^{k_3^{\max}} = -\frac{k_1}{4}, \\ d_1 = \dfrac{6\left(l_4^{k_3^{\max}}\right)^2 - k_2}{2}, \\ t = -4\left(l_4^{k_3^{\max}}\right)^3 + 4d_1\left(l_4^{k_3^{\max}}\right) - k_3. \end{cases} \tag{109}$$

The expression of $k_4^{\max}$ in the frontier situation that we are studying is

$$k_4^{\max}(k_1, k_2, k_3) = \begin{cases} \frac{k_2^2}{4} & \text{if } k_1 = 0, \\ \frac{k_1}{4}k_3 - 3\left(\frac{k_1}{4}\right)^4 & \text{if } k_2 = k_2^{\max}(k_1). \end{cases} \tag{110}$$

Figs. 6 and 7 show graphs of polynomials with $k_4^{\max}$ and several values of k_3.

Note that all the polynomials drawn in Figs. 6 and 7 have the same value at the centre and that the condition of having $k_4^{\max}$ is not deduced from their graphs but from their realizations.

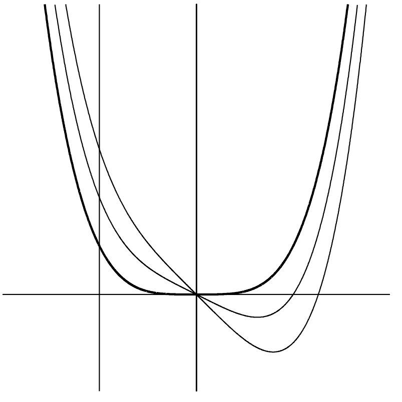


Fig. 6. $k_4^{\max}(k_1, k_2^{\max}, k_3)$.

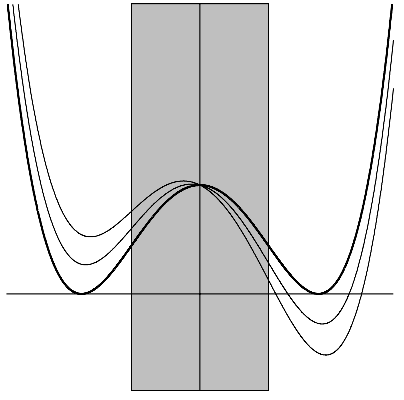


Fig. 7. $k_4^{\max}(0, k_2, k_3)$.

6.2. *The study of* $k_4^{\max}(k_1, k_2, k_3)$ *when* $k_2 > 0$

Fig. 8 shows, for different values of k_3, graphs of realizable polynomials with $k_4 = k_4^{\max}(k_1, k_2, k_3)$, k_1 and k_2 fixed. The thick continuous line represents the graph of a polynomial corresponding to $k_3^{\max}$, the dotted line represents the one corresponding to k_3^{eq} and the broken line the one corresponding to

$$k_3^{\mathrm{lt}} = k_3^{\max} - 10(x_{ri} - x_{li})^3. \tag{111}$$

This value $k_3^{\mathrm{lt}}(k_1, k_2)$, *lt* from *last tangency*, is the value of k_3 for which the right local minimum is attained at $l^{\max}(k_1, k_2)$.

As we see in Fig. 8, and as we prove in the next theorem, the polynomials corresponding to $k_4^{\max}$ have their spectral radius as double root (triple when $k_3 = k_3^{\max}$) until $k_3 = k_3^{\mathrm{lt}}$. For $k_3 > k_3^{\mathrm{lt}}$, the polynomials corresponding to $k_4^{\max}$ are characterized by having their smaller real root equal to $l^{\max}$.

The next theorem gives EBL realizations for realizable polynomials of degree 4 with $k_2 > 0$ and $k_4^{\max}$.

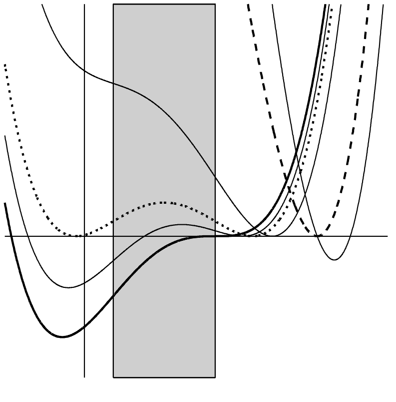


Fig. 8. $k_4^{\max}(k_1, k_2, k_3), k_2 > 0$.

Theorem 22. *Let k_1, k_2 and k_3 verify the necessary conditions (76) with $k_2 > 0$. Then the following EBL matrices have $x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{\max}(k_1, k_2, k_3)$ as characteristic polynomial:*

(1) *For $k_3^{\text{lt}} \leqslant k_3 \leqslant k_3^{\max}$ (see (111) for the definition of k_3^{lt}):*

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_1 & 1 & 0 \\ t & 0 & l_3 & 1 \\ 0 & 0 & 0 & l_4 \end{pmatrix} \quad \text{where} \quad \begin{cases} l_1 = -\frac{k_1}{4} - \frac{1}{\sqrt{6}}\sqrt{k_2^{\max} - k_2}, \\ l_3 = -\frac{k_1}{4} + \frac{2-\delta}{\sqrt{6}}\sqrt{k_2^{\max} - k_2}, \\ l_4 = -\frac{k_1}{4} + \frac{\delta}{\sqrt{6}}\sqrt{k_2^{\max} - k_2}, \\ d_1 = \frac{(\delta+1)(3-\delta)}{6}(k_2^{\max} - k_2), \\ t = \frac{\sqrt{6}}{9}(\delta + 1)(\delta - 1)^2(k_2^{\max} - k_2)^{\frac{3}{2}}, \end{cases} \tag{112}$$

where δ is the largest real root of $x^3 - 3x + 2k_3^*$ with $k_3^* = \frac{k_3 - k_3^{\text{eq}}}{k_3^{\max} - k_3^{\text{eq}}}$, that is,

$$\delta = \begin{cases} 2\cos\left(\frac{1}{3}\arccos(-k_3^*)\right) & \text{if } -k_3^{\max} + 2k_3^{\text{eq}} \leqslant k_3 \leqslant k_3^{\max}, \\ \sqrt[3]{-k_3^* + \sqrt{(k_3^*)^2 - 1}} + \sqrt[3]{-k_3^* - \sqrt{(k_3^*)^2 - 1}} & \text{if } k_3^{\text{lt}} \leqslant k_3 \leqslant -k_3^{\max} + 2k_3^{\text{eq}}. \end{cases} \tag{113}$$

Therefore $k_4^{\max}(k_1, k_2, k_3) = l_4(l_3(l_1^2 - d_1) + t)$.

(2) *For $k_3 \leqslant k_3^{\text{lt}}$:*

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ 0 & l_1 & 1 & 0 \\ t & 0 & l_1 & 1 \\ 0 & 0 & 0 & l_4 \end{pmatrix} \quad \text{where} \quad \begin{cases} l_1 = -\frac{k_1}{4} - \frac{1}{\sqrt{6}}\sqrt{k_2^{\max} - k_2}, \\ l_4 = l^{\max}(k_1, k_2), \\ t = \frac{32}{3\sqrt{6}}\left(\frac{3}{8}k_1^2 - k_2\right)^{3/2} + k_3^{\text{lt}} - k_3. \end{cases} \tag{114}$$

Therefore $k_4^{\max}(k_1, k_2, k_3) = l_4(l_1^3 + t)$.

Remark 23. If $k_3 = k_3^{\text{lt}}$ then $\delta = 3$ and the realization given in (112) is equal to the realization given in (114).

Proof. (1) The matrix is nonnegative because $l_1 = x_{li} > 0$ when $k_2 > 0$ and $\delta \in [1, 3]$. To see that this matrix has a characteristic polynomial with $k_4^{\max}(k_1, k_2, k_3)$ it is enough to see that l_4 is a double root. As $l_4 > x_{ri}$ we are at the right local minimum and therefore $k_4 = k_4^{\max}(k_1, k_2, k_3)$, see Corollary 2.

(2) Again, the matrix is nonnegative because $l_1 = x_{li} > 0$ when $k_2 > 0$. Let $Q(x)$ be the characteristic polynomial of the matrix (114) for $k_3 = k_3^{\text{lt}}$. This polynomial verifies $Q(x) > 0$, $\forall x \in (-\infty, l^{\max})$, because Q has a double root where the right local minimum is attained. Theorem 21 assures that the realizations given for $k_3 < k_3^{\text{lt}}$ have characteristic polynomials with $k_4^{\max}$, because at $l^{\max}$ they preserve the value attained by $Q(x)$. □

6.3. *The study of $k_4^{\max}(k_1, k_2, k_3)$ when $k_2 \leqslant 0$*

Fig. 9 shows, for different values of k_3, graphs of realizable polynomials with $k_4^{\max}$, $k_2 < 0$ and k_1 and k_2 fixed. The thick continuous line represents the graph of a polynomial corresponding to $k_3^{\max}$, which is characterized by having ρ and $-\rho$ as roots, where ρ is its spectral radius. This feature is kept until ρ becomes a double root. This situation is represented in Fig. 9 with a dash-dotted line curve and corresponds to

$$k_3^{trlm}(k_1, k_2) = k_3^{\text{eq}}(k_1, k_2) + \frac{k_1^2}{4}\sqrt{\frac{k_1^2}{4} - 2k_2}, \tag{115}$$

$trlm$ from *tangency at the right local minimum* (see (83) for the definition of k_3^{eq}). The existence of a double root at the spectral radius holds until the graph drawn with a broken line, which corresponds to

$$k_3^{lmk_1}(k_1, k_2) = k_1^3 + 2k_1k_2, \tag{116}$$

lmk_1 from *local minimum attained at* $-k_1$. For smaller values of k_3 the graphs of the polynomials are characterized by having $l^{\max} = -k_1$, see (89), as the smallest real root.

The features described above are completely general for $k_2 < 0$ and $k_3^{\text{eq}} \leqslant k_3 \leqslant k_3^{\max}$. For $k_3 < k_3^{\text{eq}}$, depending on which region represented in Fig. 10 the pair (k_1, k_2) belongs to, there

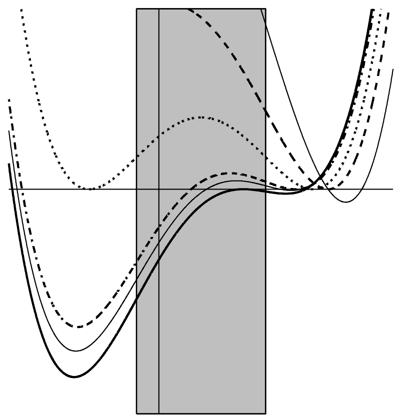


Fig. 9. $k_4 = k_4^{\max}(k_1, k_2, k_3)$.

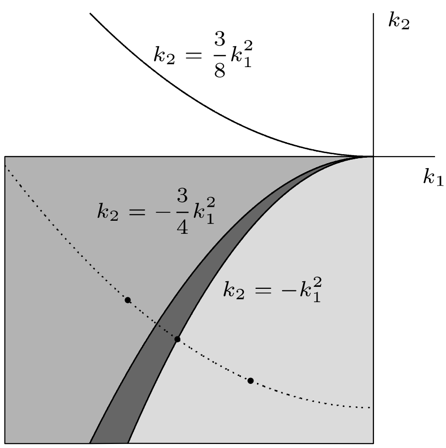


Fig. 10. Regions of (k_1, k_2).

are three models of behaviour. The model already commented corresponds to $k_2 \geqslant -\frac{3}{4}k_1^2$, that is, the shaded region from Fig. 10 bordering with the x-axis. If (k_1, k_2) belongs to either of the other two regions, then the corresponding balanced polynomial attains its local right minimum at a value larger than $l^{\max}$. This means that the tangency cannot be attained at the right local minimum, preserving the realizability for $k_3 < k_3^{\text{eq}}$, as it would have non real complex roots and a real double root in ρ larger than $l^{\max}$, see Lemma 20. Therefore, there is a transitory situation for values of k_3 between k_3^{eq} and the one corresponding to the polynomial drawn with the broken line on Figs. 11 and 12. For smaller k_3, all the polynomials with $k_4^{\max}$ meet at $x = l_{\max}$, that is, at $x = -k_1$. The value of these polynomials at $x = -k_1$ is 0 when (k_1, k_2) is in the narrow region of Fig. 10, and it is positive when (k_1, k_2) is in the shaded region bordering with the y-axis of the Fig. 10.

The pairs (k_1, k_2) used in Figs. 9, 11 and 12 correspond to the points represented on Fig. 10. Note that these three points are on the parabola $k_2^{\max} - k_2 = c$ and so the distance between the inflexion points is the same in the three cases.

6.3.1. *From $k_3^{\max}(k_1, k_2)$ to $k_3^{\text{eq}}(k_1, k_2)$*

We shall now prove that the polynomials described above for the range $k_3^{\text{eq}} \leqslant k_3 \leqslant k_3^{\max}$ correspond to $k_4^{\max}(k_1, k_2, k_3)$.

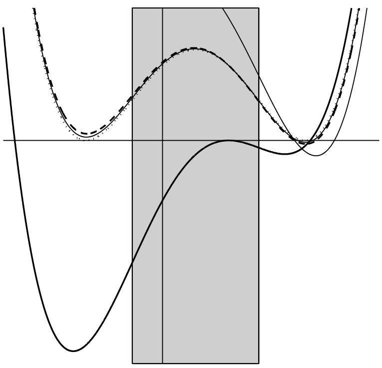


Fig. 11. $k_4 = k_4^{\max}(k_1, k_2, k_3)$, $-k_1^2 \leqslant k_2 < -\frac{3}{4}k_1^2$.

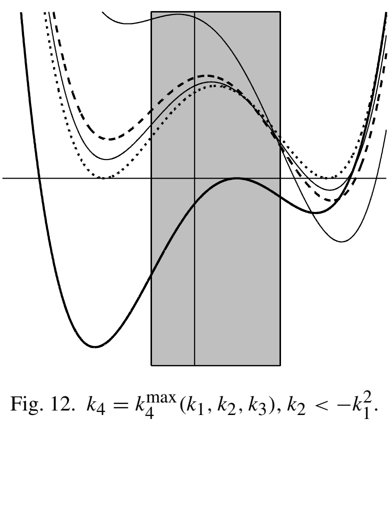


Fig. 12. $k_4 = k_4^{\max}(k_1, k_2, k_3)$, $k_2 < -k_1^2$.

Theorem 24. *Let k_1, k_2 and k_3 verify the necessary conditions (76) with $k_2 \leqslant 0$ and $k_3 \geqslant k_3^{\mathrm{eq}}(k_1,k_2)$. Then the following EBL matrices have $x^4+k_1x^3+k_2x^2+k_3x+k_4^{\max}(k_1,k_2,k_3)$ as characteristic polynomial*:

(1) *For $k_3^{trlm} \leqslant k_3 \leqslant k_3^{\max}$ (see (115) for the definition of k_3^{trlm})*:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ d_1 & 0 & 1 & 0 \\ 0 & 0 & l_4 & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix} \quad \textit{where} \begin{cases} l_4 = -\frac{k_1}{2}, \\ d_1 = \frac{k_3}{k_3^{\max}}\left(\frac{k_1^2}{4}-k_2\right), \\ d_3 = \left(1-\frac{k_3}{k_3^{\max}}\right)\left(\frac{k_1^2}{4}-k_2\right). \end{cases} \tag{117}$$

Therefore $k_4^{\max}(k_1,k_2,k_3) = d_1(d_3-l_4^2)$.

(2) *For $k_3^{\mathrm{eq}} \leqslant k_3 \leqslant k_3^{trlm}$*:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ 0 & 0 & l_4 & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix} \quad \textit{where} \begin{cases} l_2 = -\frac{k_1}{2}-\sqrt{\delta}, \\ l_4 = -\frac{k_1}{4}+\frac{\sqrt{\delta}}{2}, \\ d_1 = m\left(\sqrt{\delta}+m+\frac{k_1}{2}\right), \\ d_3 = \left(\frac{\sqrt{\delta}}{2}-m-\frac{k_1}{4}\right)^2, \end{cases} \tag{118}$$

where δ is

$$\delta = \frac{2(k_3-k_3^{\mathrm{eq}})}{4m+k_1} \tag{119}$$

and m is the x-coordinate where $x^4+k_1x^3+k_2x^2+k_3x$ attains its right local minimum, i.e.,

$$m = -\frac{k_1}{4}+\frac{\sqrt{6}}{3}\sqrt{k_2^{\max}-k_2}\cos\left(\frac{1}{3}\arccos\left(-\frac{3\sqrt{3}}{\sqrt{8}}\left(\frac{k_3-k_3^{\mathrm{eq}}}{(k_2^{\max}-k_2)^{\frac{3}{2}}}\right)\right)\right). \tag{120}$$

Therefore $k_4^{\max}(k_1,k_2,k_3) = d_1(d_3-l_4^2)$.

Remark 25. When $k_3 = k_3^{trlm}$ the two realizations given are equal.

Proof. (1) The matrix is nonnegative because $k_2 \leqslant 0$ and $k_3 \geqslant k_3^{\mathrm{eq}} \geqslant 0$. Its eigenvalues are

$$\pm\sqrt{\frac{k_3}{k_3^{\max}}}\sqrt{\frac{k_1^2}{4}-k_2}, \quad -\frac{k_1}{2} \pm \sqrt{\left(1-\frac{k_3}{k_3^{\max}}\right)\left(\frac{k_1^2}{4}-k_2\right)}. \tag{121}$$

The value of $k_3/k_3^{\max}$ is 1 when $k_3 = k_3^{\max}$ and decreases with k_3 until the spectral radius $\rho = \sqrt{\frac{k_3}{k_3^{\max}}}\sqrt{\frac{k_1^2}{4}-k_2}$ becomes a double root when $k_3 = k_3^{trlm}$, that is, k_3^{trlm} solves

$$\sqrt{\frac{k_3}{k_3^{\max}}}\sqrt{\frac{k_1^2}{4}-k_2} = -\frac{k_1}{2}+\sqrt{\left(1-\frac{k_3}{k_3^{\max}}\right)\left(\frac{k_1^2}{4}-k_2\right)}. \tag{122}$$

Then the characteristic polynomial of (117) verifies Proposition 4 and so its value at zero is $k_4^{\max}$.

(2) Let us see that the matrix is nonnegative. Note that δ decreases when k_3 decreases (the x-coordinate where the right local minimum is attained grows when the derivate decreases).

Thus, the maximum value of δ corresponds to k_3^{trlm} and the corresponding value of the right local minimum is $\frac{1}{4}\left(-k_1+\sqrt{k_1^2-k_2}\right)$. Using this expression it can be seen that $-k_1^2/4$ is the maximum value of δ and so $l_2 \geqslant 0$. Finally, $d_1 \geqslant 0$ because $m > x_{ri} > -k_1/2$. Now the result follows because the matrix (118) has m as double eigenvalue. □

6.3.2. The case $k_3 < k_3^{\text{eq}}(k_1, k_2)$ with $-\frac{3}{4}k_1^2 \leqslant k_2 \leqslant 0$

Let $m_{\text{eq}}(k_1, k_2)$ be the x-coordinate where the right local minimum of the balanced polynomial $x^4 + k_1x^3 + k_2x^2 + k_3^{\text{eq}}x + k_4^{\max}(k_1, k_2, k_3^{\text{eq}})$ is attained, that is,

$$m_{\text{eq}}(k_1, k_2) = -\frac{k_1}{4} + \frac{1}{\sqrt{2}}\sqrt{k_2^{\max} - k_2}. \tag{123}$$

The condition $-\frac{3}{4}k_1^2 \leqslant k_2$ assures that $m_{\text{eq}} \leqslant -k_1$. This allows $k_4^{\max}$ to be attained with tangency at the right local minimum for some values of k_3 smaller than k_3^{eq}. Exactly, for all k_3 corresponding to a right local minimum at an x-coordinate smaller than or equal to $-k_1$. It can be seen that $k_3^{lmk_1}$, see (116), is the smallest k_3 verifying this. When $k_3 < k_3^{lmk_1}$ the corresponding polynomial with $k_4^{\max}$ cannot attain tangency at the right local minimum because the existence of non real complex roots implies reducibility and then the local minimum cannot be attained at l_4.

Theorem 26. *Let k_1, k_2 and k_3 verify the necessary conditions (76) with $-\frac{3}{4}k_1^2 \leqslant k_2 \leqslant 0$ and $k_3 \leqslant k_3^{\text{eq}}(k_1, k_2)$. Then the following EBL matrices have $x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{\max}(k_1, k_2, k_3)$ as characteristic polynomial*:

(1) *For $k_3^{lmk_1} \leqslant k_3 \leqslant k_3^{\text{eq}}$ (see (116) for the definition of $k_3^{lmk_1}$)*:

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_1 & 1 & 0 \\ t & 0 & l_1 & 1 \\ 0 & 0 & 0 & l_4 \end{pmatrix} \quad \textit{where} \begin{cases} l_1 = -\frac{k_1+m}{3}, \\ l_4 = m, \\ d_1 = \frac{1}{3}(k_1^2 - k_1m - 3k_2 - 2m^2), \\ t = -\frac{(k_1+4m)}{27}(2k_1^2 - 11k_1m - 9k_2 - 22m^2), \end{cases} \tag{124}$$

and m is the x-coordinate where $x^4 + k_1x^3 + k_2x^2 + k_3x$ attains its right local minimum, i.e.,

$$m = -\frac{k_1}{4} + \frac{\sqrt{6}}{3}\sqrt{k_2^{\max} - k_2}\cos\left(\frac{1}{3}\arccos\left(-\frac{3\sqrt{3}}{\sqrt{8}}\left(\frac{k_3 - k_3^{\text{eq}}}{(k_2^{\max} - k_2)^{\frac{3}{2}}}\right)\right)\right). \tag{125}$$

Therefore $k_4^{\max}(k_1, k_2, k_3) = l_4(t + l_1^3 - l_1d_1)$.

(2) *For $k_3 \leqslant k_3^{lmk_1}$*:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ d_1 & 0 & 1 & 0 \\ t & 0 & 0 & 1 \\ 0 & 0 & 0 & l_4 \end{pmatrix} \quad \textit{where} \begin{cases} l_4 = -k_1, \\ d_1 = -k_2, \\ t = k_1k_2 - k_3. \end{cases} \tag{126}$$

Therefore $k_4^{\max}(k_1, k_2, k_3) = tl_4$.

Remark 27. When $k_3 = k_3^{lmk_1}$ the two realizations given are equal.

Proof. (1) The matrix (124) has m as double eigenvalue. Let us see that this matrix is nonnegative. The diagonal elements are nonnegative because $m \in [m_{\text{eq}}, -k_1]$ for the range of k_3 considered. The element d_1 is a polynomial of degree 2 in m with roots

$$-\frac{k_1}{4} \pm \sqrt{\frac{3}{2}}\sqrt{k_2^{\max} - k_2}, \tag{127}$$

and the nonnegative character of d_1 follows from

$$-\frac{k_1}{4} - \sqrt{\frac{3}{2}}\sqrt{k_2^{\max} - k_2} \leqslant m_{\text{eq}} \leqslant -k_1 \leqslant -\frac{k_1}{4} + \sqrt{\frac{3}{2}}\sqrt{k_2^{\max} - k_2}. \tag{128}$$

Finally, let us see that $t \geqslant 0$. Since $k_1 + 4m > 0$, the result follows if $2k_1^2 - 11k_1m - 9k_2 - 22m^2 \leqslant 0$, but the roots of this polynomial in m are

$$-\frac{k_1}{4} \pm \frac{3}{\sqrt{22}}\sqrt{k_2^{\max} - k_2} \tag{129}$$

and the values of m for the range of k_3 considered are greater than the greatest of these roots because

$$m \geqslant m_{\text{eq}} > -\frac{k_1}{4} + \frac{3}{\sqrt{22}}\sqrt{k_2^{\max} - k_2}. \tag{130}$$

(2) The nonnegative character of the matrix is a consequence of $k_3 \leqslant k_3^{lmk_1} \leqslant k_1k_2$, see (116). When $k_3 = k_3^{lmk_1}$ the result follows because the graph of the characteristic polynomial of the matrix is tangent to the x-axis at the unique real root. For other values of k_3 the result follows from Theorem 21. Note that for these polynomials $P(l_4) = P(-k_1) = 0$. □

6.3.3. The case $k_3 < k_3^{\text{eq}}(k_1, k_2)$ with $k_2 < -\frac{3}{4}k_1^2$

The condition $k_2 < -\frac{3}{4}k_1^2$ assures that $m_{\text{eq}} > -k_1$. This means that no $k_3 < k_3^{\text{eq}}$ has a corresponding $k_4^{\max}$ with tangency at the right local minimum, because this implies non real complex roots, reducible realization and a value for l_4 greater than $-k_1$.

We shall now obtain necessary conditions on the EBL realization patterns with $k_4^{\max}$.

Lemma 28. *Let k_1, k_2 and k_3 verify the necessary conditions (76) with $k_2 < -\frac{3}{4}k_1^2$ and $k_3 < k_3^{\text{eq}}(k_1, k_2)$. Then $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{\max}(k_1, k_2, k_3)$ has non real complex roots.*

Proof. Let us consider the characteristic polynomials of the following matrices:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ d_1 & l_4 & 1 & 0 \\ t & 0 & 0 & 1 \\ 0 & 0 & d_1 & l_4 \end{pmatrix} \quad \text{where} \begin{cases} l_4 = -\frac{k_1}{2}, \\ d_1 = \frac{1}{2}\left(\frac{k_1^2}{4} - k_2\right), \\ t = k_3^{\text{eq}} - k_3. \end{cases} \tag{131}$$

When $k_3 = k_3^{\text{eq}}$ this polynomial corresponds to $k_4^{\max}$ (whose graph is drawn with a dotted line in Fig. 13). For $k_3 < k_3^{\text{eq}}$ all the polynomials meet the one considered for $k_3 = k_3^{\text{eq}}$ at $x = -k_1/2$, see Fig. 13, and this is the only meeting point because the graphs of the derivatives of these polynomials are parallel. This assures that, for $k_3 < k_3^{\text{eq}}$, these polynomials have non real complex roots, and the same is true for any polynomial with equal k_1, k_2 and k_3 and greater independent term. □

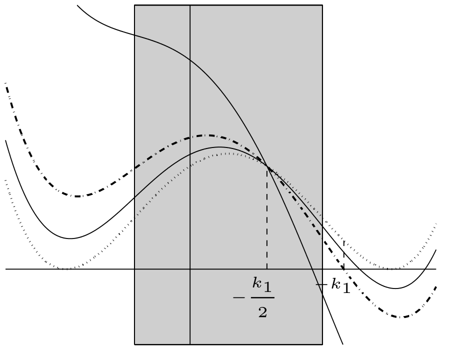


Fig. 13. Graphs of the characteristic polynomials of (131).

Theorem 29. *Let k_1, k_2 and k_3 verify the necessary conditions (76) with $k_2 < -\frac{3}{4}k_1^2$ and $k_3 < k_3^{\mathrm{eq}}(k_1, k_2)$. Let $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{\max}(k_1, k_2, k_3)$.*

(1) *If P has a reducible realization, then:*

(1.1) $k_3 \leqslant k_3^{\mathrm{eq}}(k_1, k_2) + \frac{1}{2k_1}\left(\frac{3}{4}k_1^2 + k_2\right)^2$,

(1.2) $P(-k_1) = 0$ *and* $k_4^{\max}(k_1, k_2, k_3) = k_1(k_3 - k_1k_2)$.

(2) *If P does not admit a reducible realization, then $P(-k_1) > 0$.*

Proof. (1.1) Let us consider the characteristic polynomial of the matrix (131) for $k_3 = k_3^{\mathrm{eq}} + \frac{1}{2k_1}\left(\frac{3}{4}k_1^2 + k_2\right)^2$. This polynomial, whose graph is drawn with a dashpointed line in Fig. 13, has $-k_1$ as the lowest real root. For values of k_3 greater than this (and smaller than k_3^{eq}) the two real roots of the characteristic polynomial of the matrix (131) are greater than $-k_1$, and the same happens for the polynomials corresponding to $k_4^{\max}$. Lemma 20 assures that the only possible realizations are irreducible.

(1.2) If P admits a reducible realization, then P has a real root lower than or equal to $-k_1$ (allowing this root to be the greatest diagonal element, l_4). Then, the best option among the reducible ones (the one with the largest k_4) is the one with a root at $-k_1$. A polynomial with k_1, k_2 and k_3 verifying the conditions of the theorem and with a root at $-k_1$ has $k_1(k_3 - k_1k_2)$ as independent term, and it is realizable by the nonnegative matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ d_1 & 0 & 1 & 0 \\ t & 0 & 0 & 1 \\ 0 & 0 & 0 & l_4 \end{pmatrix} \quad \textit{where} \begin{cases} l_4 = -k_1, \\ d_1 = -k_2, \\ t = k_1k_2 - k_3. \end{cases} \tag{132}$$

(2) The polynomial $x^4 + k_1x^3 + k_2x^2 + k_3x + k_1(k_3 - k_1k_2)$, realized by (132), guarantees that $P(-k_1) \geqslant 0$. The result follows from (1.2). □

The previous result finishes our study of the reducible realizations corresponding to $k_4^{\max}$. Therefore, in what follows we concentrate our attention on describing the irreducible ones, where we know the maximum k_4 is attained, at least for values of k_3 close to k_3^{eq}.

Theorem 30. *Let k_1, k_2 and k_3 verify the necessary conditions* (76) *with* $k_2 < -\frac{3}{4}k_1^2$, $k_3 < k_3^{\mathrm{eq}}(k_1, k_2)$ *and let* $P(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{\max}(k_1, k_2, k_3)$ *be a polynomial that does not admit a reducible realization. Then every EBL realization of* $P(x)$

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ t & d_2 & l_3 & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix} \tag{133}$$

must verify:

(1) $t > 0$ *and* $d_3 > 0$,
(2) $l_4 > -k_1/2$,
(3) $d_2 = 0$,
(4) $d_1 - d_3 = (l_4 - l_1)(l_4 - l_2)$,
(5) $l_3 = 0$,
(6) l_1 *can be taken as zero*,
(7) $l_4 \leqslant \min\left\{\frac{\sqrt{k_1^2-3k_2}-k_1}{3}, -k_1\right\}$.

Therefore

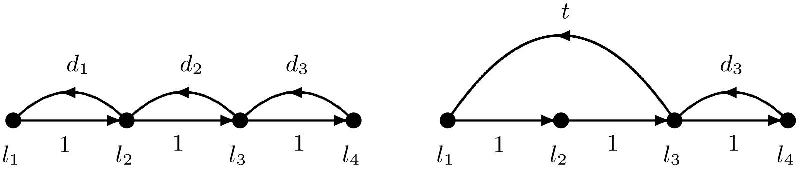
$$k_4^{\max}(k_1, k_2, k_3) = \max_{-\frac{k_1}{2} < l_4 \leqslant \min\left\{\frac{\sqrt{k_1^2-3k_2}-k_1}{3}, -k_1\right\}} k_4^{ir}(l_4, k_1, k_2, k_3), \tag{134}$$

where

$$k_4^{ir}(l_4, k_1, k_2, k_3) = \frac{5l_4^4}{4} + 2k_1l_4^3 + l_4^2\left(k_1^2 + \frac{k_2}{2}\right) + l_4(k_1k_2 - k_3) + \frac{k_2^2}{4}, \tag{135}$$

ir *from* irreducible realization.

Proof. (1) Irreducible matricial realizations are equivalent to strongly connected digraphs, and for the EBL digraphs considered this implies $d_1d_2d_3 \neq 0$ or $td_3 \neq 0$.



In both cases $d_3 > 0$. Now $t > 0$, otherwise P would have a symmetric realization, see (104), and it goes against the fact that P has non real complex roots (see Lemma 28).

(2) Assume $l_4 \leqslant -k_1/2$ and let $Q(x)$ be the characteristic polynomial of a matrix like (131) and $Q'(x) = P'(x)$ (*i.e.*, the coefficients of degrees 3, 2 and 1 of $P(x)$ and $Q(x)$ are equal). Let us consider the minimum weight of the 3-cycles of the EBL realizations of $Q(x)$ and $P(x)$, that is,

$$t_{\min} = \min\{k_3^{\mathrm{eq}} - k_3, t\}. \tag{136}$$

Now let $Q_0(x)$ and $P_0(x)$ be the characteristic polynomials obtained from the realizations of $Q(x)$ and $P(x)$ respectively on deminishing the entry (3, 1) by $t_{\min}$, that is

$$\begin{aligned} Q_0(x) &= Q(x) + t_{\min}x - t_{\min}\left(-\frac{k_1}{2}\right), \\ P_0(x) &= P(x) + t_{\min}x - t_{\min}l_4. \end{aligned} \tag{137}$$

Subtracting these equalities we get

$$P_0(x) - Q_0(x) = P(x) - Q(x) + t_{\min}\left(-\frac{k_1}{2} - l_4\right) \geqslant 0. \tag{138}$$

If $t_{\min} = k_3^{\mathrm{eq}} - k_3$, then the realization that we have for Q_0 has no 3-cycles, *i.e.*, it is balanced. As the graph of P_0 is above the graph of Q_0, and P_0 is realizable, it should coincide with the graph of Q_0. This means that $P - Q = 0$, which is impossible because P does not admit a reducible realization and Q does.

If $t_{\min} < k_3^{\mathrm{eq}} - k_3$, then $t_{\min} = t$ and the realization of P_0 has no 3-cycles. It thus admits symmetric realization, *i.e.*, it has four real roots. This is impossible because its graph is above the graph of Q_0, and P_0 has a realization of the form (131) with the entry (3, 1) positive and so P_0 has non real complex roots.

(3) Assume $d_2 > 0$. Consider the nonnegative matrix, for a sufficiently small $\varepsilon > 0$,

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 + \varepsilon & l_2 & 1 & 0 \\ t + \varepsilon(l_3 - l_1) & d_2 - \varepsilon & l_3 & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix} \tag{139}$$

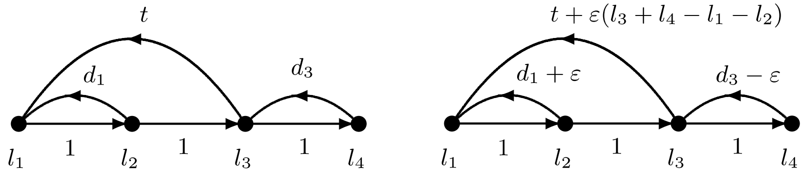
whose characteristic polynomial is $P_\varepsilon(x) = x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{\max} + \varepsilon d_3$. We get a contradiction because $P_\varepsilon(0) > k_4^{\max}$.

(4) Firstly, let us see that $d_1 \geqslant d_3$. Otherwise, the nonnegative matrix

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_3 & l_2 & 1 & 0 \\ t + (d_3 - d_1)(l_3 + l_4 - l_1 - l_2) & 0 & l_3 & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix} \tag{140}$$

has a characteristic polynomial, $x^4 + k_1x^3 + k_2x^2 + k_3x + k_4^{\max} + (d_3 - d_1)(l_4 - l_1)(l_4 - l_2)$, with an independent term larger than $k_4^{\max}$.

As $d_1 \geqslant d_3 > 0$ we can consider the following digraphs



whose characteristic polynomials have the same derivative and the difference between their independent terms is

$$\varepsilon^2 + \varepsilon((d_3 - d_1) + (l_4 - l_1)(l_4 - l_2)). \tag{141}$$

The characteristic polynomial of the left digraph is $P(x)$ and then the above value is nonnegative for ε in a neighbourhood of 0 if and only if

$$(d_3 - d_1) + (l_4 - l_1)(l_4 - l_2) = 0. \tag{142}$$

(5) Assume $l_3 > 0$. Let $Q(x)$ be the characteristic polynomial of the nonnegative matrix, for a sufficiently small $\varepsilon > 0$,

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_3 + (l_4 - l_1)(l_4 - l_2) - \varepsilon(\varepsilon + l_2 - l_3) & l_2 + \varepsilon & 1 & 0 \\ t + \varepsilon(\varepsilon + l_1 + l_2 - l_3 - l_4)(\varepsilon - l_3 + l_4) & 0 & l_3 - \varepsilon & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix}. \tag{143}$$

It can be seen that, for a sufficiently small $\varepsilon > 0$, $Q(x) - P(x) = -d_3\varepsilon(\varepsilon + l_1 + l_2 - l_3 - l_4) > 0$ which contradicts $P(0) = k_4^{\max}$.

(6) l_1 can be taken as zero because the following matrices have $P(x)$ as characteristic polynomial:

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_3 + (l_4 - l_1)(l_4 - l_2) & l_2 & 1 & 0 \\ t & 0 & l_3 & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ d_3 + l_4(l_4 - (l_1 + l_2)) & l_2 + l_1 & 1 & 0 \\ t & 0 & l_3 & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix}. \tag{144}$$

(7) From the matrix realization of $P(x)$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ d_3 + l_4(l_4 - l_2) & l_2 & 1 & 0 \\ t & 0 & 0 & 1 \\ 0 & 0 & d_3 & l_4 \end{pmatrix} \tag{145}$$

we obtain the following relations

$$\begin{aligned} l_2 &= -k_1 - l_4, \\ d_3 &= -\frac{3}{2}l_4^2 - k_1 l_4 - \frac{k_2}{2}. \end{aligned} \tag{146}$$

When $l_4 \in (-\frac{k_1}{2}, -k_1]$, we have $d_3 \geqslant 0$ if and only if

$$l_4 \leqslant \frac{\sqrt{k_1^2 - 3k_2} - k_1}{3}. \tag{147}$$

This restriction is only relevant when $\frac{\sqrt{k_1^2 - 3k_2} - k_1}{3} < -k_1$, *i.e.*, when $-k_1^2 < k_2$. Otherwise (146) is verified for all $l_4 \in (\frac{-k_1}{2}, -k_1]$.

Finally, as a consequence of all the conditions proved, the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{l_4^2-k_2}{2} & -k_1-l_4 & 1 & 0 \\ k_3^{\mathrm{eq}}-k_3+2\left(\frac{k_1}{4}+l_4\right)\left(\frac{k_1}{2}+l_4\right)^2 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3}{2}l_4^2-k_1l_4-\frac{k_2}{2} & l_4 \end{pmatrix} \tag{148}$$

is a realization of P which shows (134). □

Remark 31. Note that

$$\max_{-\frac{k_1}{2}<l_4\leqslant\min\left\{\frac{\sqrt{k_1^2-3k_2}-k_1}{3},-k_1\right\}} k_4^{ir}(l_4,k_1,k_2,k_3) = \max_{-\frac{k_1}{2}\leqslant l_4\leqslant\min\left\{\frac{\sqrt{k_1^2-3k_2}-k_1}{3},-k_1\right\}} k_4^{ir}(l_4,k_1,k_2,k_3), \tag{149}$$

because $\frac{\partial}{\partial l_4}k_4^{ir}\left(-\frac{k_1}{2},k_1,k_2,k_3\right)>0$.

Remark 32. When P admits a reducible realization, we know that $k_4^{\max}(k_1,k_2,k_3)=k_1(k_3-k_1k_2)$, so we can assure that

$$k_4^{\max}(k_1,k_2,k_3)=\max\left\{k_1(k_3-k_1k_2),\max_{-\frac{k_1}{2}\leqslant l_4\leqslant\min\left\{\frac{\sqrt{k_1^2-3k_2}-k_1}{3},-k_1\right\}} k_4^{ir}(l_4,k_1,k_2,k_3)\right\}. \tag{150}$$

To complete the study of $k_4^{\max}$ we will distinguish the two situations $-k_1^2<k_2<-\frac{3}{4}k_1^2$ and $k_2\leqslant -k_1^2$.

Theorem 33. *Let k_1, k_2 and k_3 verify the necessary conditions (76) with $-k_1^2<k_2<-\frac{3}{4}k_1^2$ and $k_3<k_3^{\mathrm{eq}}(k_1,k_2)$. Then*

$$k_4^{\max}(k_1,k_2,k_3)=\begin{cases} k_4^{ir}(l_4^m(k_1,k_2,k_3),k_1,k_2,k_3) & \text{if } k_3^*<k_3,\\ -k_1(k_1k_2-k_3) & \text{if } k_3\leqslant k_3^*,\end{cases} \tag{151}$$

where $l_4^m(k_1,k_2,k_3)$ is the x-coordinate where $k_4^{ir}(l_4,k_1,k_2,k_3)$ attains its local maximum as a function of l_4, that is,

$$l_4^m(k_1,k_2,k_3) = -\frac{2k_1}{5}+\frac{2}{5\sqrt{3}}\sqrt{2k_1^2-5k_2}\,\sin\left(\frac{1}{3}\arcsin\left(\frac{3\sqrt{3}(-4k_1^3+15k_1k_2-25k_3)}{2(2k_1^2-5k_2)^{3/2}}\right)\right) \tag{152}$$

and k_3^* is the greatest value of k_3 that verifies the equation

$$k_4^{ir}(l_4^m(k_1,k_2,k_3),k_1,k_2,k_3)=k_1(k_3-k_1k_2). \tag{153}$$

Proof. When $-k_1^2 < k_2 < -\frac{3}{4}k_1^2$ we have $l_4 \leqslant \frac{\sqrt{k_1^2-3k_2}-k_1}{3} < -k_1$, so the expression for $k_4^{\max}$ given in (150) is

$$k_4^{\max}(k_1,k_2,k_3) = \max\left\{k_1(k_3-k_1k_2), \max_{-\frac{k_1}{2}\leqslant l_4 \leqslant \frac{\sqrt{k_1^2-3k_2}-k_1}{3}} k_4^{ir}(l_4,k_1,k_2,k_3)\right\}. \tag{154}$$

The function $k_4^{ir}(l_4,k_1,k_2,k_3)$, see (135), is a polynomial of degree 4 in l_4 and cannot attain the maximum that appears in (154) at $l_4 = -k_1/2$ (see the Remark 31). Then

$$\max_{-\frac{k_1}{2}\leqslant l_4 \leqslant \min\left\{\frac{\sqrt{k_1^2-3k_2}-k_1}{3},-k_1\right\}} k_4^{ir}(l_4,k_1,k_2,k_3)$$
$$= \max\left\{k_4^{ir}(l_4^m,k_1,k_2,k_3), k_4^{ir}\left(\frac{\sqrt{k_1^2-3k_2}-k_1}{3},k_1,k_2,k_3\right)\right\}. \tag{155}$$

But if the maximum is attained at the extreme righthand side of the interval, the matrix (148) will have $d_3 = 0$, that is, it would be reducible and the value of $k_4^{\max}$ will then be $k_1(k_3-k_1k_2)$. Then

$$k_4^{\max}(k_1,k_2,k_3) = \max\{k_1(k_3-k_1k_2), k_4^{ir}(l_4^m,k_1,k_2,k_3)\}. \tag{156}$$

An analysis of the behaviour of k_4^{ir} as a function of l_4, for ever decreasing values of k_3, shows that the extreme absolute of this function in the interval under study is finally attained at the extreme right side of the interval (at $l_4 = \frac{\sqrt{k_1^2-3k_2}-k_1}{3}$) since, for a small enough k_3, the function k_4^{ir} ends by being an increasing function of l_4 (the local maximum disappears). As the value of k_4^{ir} at the right extreme of the interval is smaller than the one corresponding to a reducible realization, the existence of a k_3^* verifying (153) is assured. The Theorem 21 now guarantees that if $k_4^{\max}$ has been attained for a value of k_3 with a realization as (132), for smaller values of k_3, the root at $l^{\max}$ will be maintained, that is, the $k_4^{\max}$ will still correspond to a realization of type (132). □

Theorem 34. *Let k_1, k_2 and k_3 verify the necessary conditions (76) with $k_2 \leqslant -k_1^2$ and $k_3 < k_3^{\mathrm{eq}}(k_1,k_2)$. Let*

$$k_3^{cc} = -\frac{\sqrt{6}}{225}(-k_1^2-5k_2)^{\frac{3}{2}} - \frac{k_1}{25}(7k_1^2-10k_2), \tag{157}$$

cc *from* common cut. *Then*

$$k_4^{\max}(k_1,k_2,k_3) = \begin{cases} k_4^{ir}(l_4^m(k_1,k_2,k_3),k_1,k_2,k_3) & \text{if } k_3^{cc} < k_3, \\ k_4^{ir}(-k_1,k_1,k_2,k_3) & \text{if } k_3 \leqslant k_3^{cc}, \end{cases} \tag{158}$$

where $l_4^m(k_1,k_2,k_3)$ is the x-coordinate where $k_4^{ir}(l_4,k_1,k_2,k_3)$ attains its local maximum as a function of l_4, see (152) for its expression.

Proof. When $k_2 \leqslant -k_1^2$, the expression of $k_4^{\max}$ given in (150) is

$$k_4^{\max}(k_1,k_2,k_3) = \max\left\{k_1(k_3-k_1k_2), \max_{-\frac{k_1}{2}\leqslant l_4 \leqslant -k_1} k_4^{ir}(l_4,k_1,k_2,k_3)\right\}. \tag{159}$$

Because

$$k_4^{ir}(-k_1, k_1, k_2, k_3) - k_1(k_3 - k_1k_2) = \frac{(k_1^2 + k_2)^2}{4} \geqslant 0 \tag{160}$$

then

$$k_4^{\max}(k_1, k_2, k_3) = \max_{-\frac{k_1}{2} \leqslant l_4 \leqslant -k_1} k_4^{ir}(l_4, k_1, k_2, k_3). \tag{161}$$

The same argument that was used in the proof of the previous theorem gives that $k_4^{\max}$ is attained at $l_4^m(k_1, k_2, k_3)$ for k_3 close to k_3^{eq}. Decreasing k_3 there is a moment when

$$k_4^{ir}(l_4^m, k_1, k_2, k_3) = k_4^{ir}(-k_1, k_1, k_2, k_3), \tag{162}$$

that is, the local maximum is attained at $-k_1$. Applying Theorem 21, for smaller values of k_3, we obtain that the polynomials with $k_4^{\max}$ meet at the point $(-k_1, \frac{(k_1^2+k_2)^2}{4})$. This corresponds to an irreducible realization with $l_4 = -k_1$. □

Example 35. If $k_1 = -1$, $k_2 = -7/5$ and $k_3 \leqslant k_3^{\max} = 33/20$, then the polynomial $x^4 - x^3 - \frac{7}{5}x^2 + k_3x + k_4^{\max}$ is always realizable by Theorem 17. Note that $k_2 = -7/5 < -k_1^2 = -1$, so if $k_3 < k_3^{\mathrm{eq}} = 33/40$ we are under the assumptions of Theorem 34, which gives the value of $k_4^{\max}$ and the matrix (148) gives an EBL realization for the polynomial. Let consider three particular values of k_3:

- $k_3 = 1 - \frac{3\sqrt{3}}{25} \in (k_3^{cc}, k_3^{\mathrm{eq}}) = \left(\frac{17}{25}, \frac{33}{40}\right)$. The polynomial

$$P(x) = x^4 - x^3 - \frac{7}{5}x^2 + \left(1 - \frac{3\sqrt{3}}{25}\right)x + \frac{229}{500}$$
$$- \frac{18\cos^2\left(\frac{\pi}{18}\right)}{125} + \frac{36\cos^4\left(\frac{\pi}{18}\right)}{125} + \frac{6\sqrt{3}}{125} + \frac{18\sin\left(\frac{\pi}{18}\right)}{125}$$

  is realized by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{51}{50} + \frac{4\sqrt{3}\sin\left(\frac{\pi}{18}\right) - 6\cos^2\left(\frac{\pi}{18}\right)}{25} & \frac{3 - 2\sqrt{3}\sin\left(\frac{\pi}{18}\right)}{5} & 1 & 0 \\ \frac{6\sqrt{3}-11}{50} + \frac{46\sqrt{3}\sin\left(\frac{\pi}{18}\right) + 6\cos^2\left(\frac{\pi}{18}\right)\left(1 - 8\sqrt{3}\sin\left(\frac{\pi}{18}\right)\right)}{125} & 0 & 0 & 1 \\ 0 & 0 & \frac{7 + 4\left(9\cos^2\left(\frac{\pi}{18}\right) - \sqrt{3}\sin\left(\frac{\pi}{18}\right)\right)}{50} & \frac{2 + 2\sqrt{3}\sin\left(\frac{\pi}{18}\right)}{5} \end{pmatrix}.$$

  For this value of k_3 the maximum of k_4^{ir}, as function of its first variable, is attained at $l_4^m = \frac{2}{5} + \frac{2}{5}\sqrt{3}\sin(\frac{\pi}{18}) < -k_1 = 1$, see (152).
- $k_3 = k_3^{cc} = \frac{17}{25}$. The polynomial $Q(x) = x^4 - x^3 - \frac{7}{5}x^2 + \frac{17}{25}x + \frac{19}{25}$ is realized by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{6}{5} & 0 & 1 & 0 \\ \frac{13}{25} & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & 1 \end{pmatrix}.$$

- $k_3 = \frac{12}{25} < k_3^{cc} = \frac{17}{25}$. The polynomial $R(x) = x^4 - x^3 - \frac{7}{5}x^2 + \frac{12}{25}x + \frac{24}{25}$ is realized by

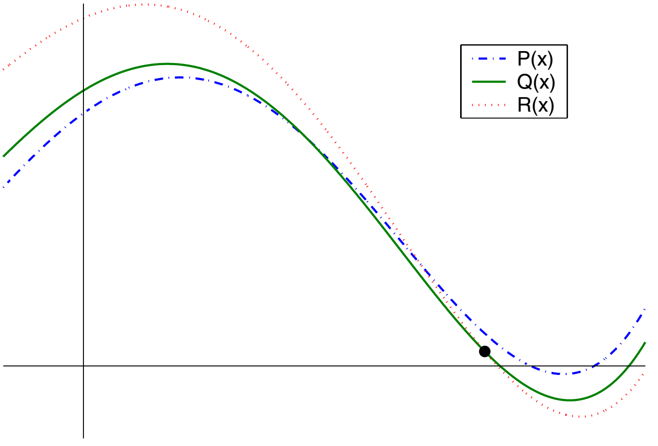


Fig. 14. Graphs of the polynomials from Example 35.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{6}{5} & 0 & 1 & 0 \\ \frac{18}{25} & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & 1 \end{pmatrix}.$$

For this value of k_3, as well as for the previous one $k_3 = k_3^{cc}$, the maximum of k_4^{ir}, as function of its first variable, is attained at $l_4 = -k_1 = 1$. Therefore, the realizations given by (148) are only different in the element (3, 1) and the polynomials $Q(x)$ and $R(x)$ have the same value at $-k_1$, which is represented by a point on Fig. 14.

Remark 36. Note that, as a result of the EBL realizations obtained for $k_4^{\max}$, it can be said that any realizable polynomial of degree 4, $x^4 + k_1x^3 + k_2x^2 + k_3x + k_4$, is realizable by an EBL matrix of the type

$$\begin{pmatrix} l_1 & 1 & 0 & 0 \\ d_1 & l_2 & 1 & 0 \\ t & 0 & l_3 & 1 \\ c & 0 & d_3 & l_4 \end{pmatrix}, \tag{163}$$

for this, it is sufficient to take the realization given in this paper corresponding to $k_4^{\max}(k_1, k_2, k_3)$ (all having $d_2 = 0$) and to put the corresponding c to go from the value of $k_4^{\max}$ to the value of k_4, *i.e.*, $c = k_4^{\max} - k_4$.

7. The case $n \leqslant 2p + 1$ when $k_1 = \cdots = k_{p-1} = 0$, $p \geqslant 2$

The techniques developed in this paper and the Newton identities, allow us to extend the already known result of obtaining necessary and sufficient conditions for a family of five complex numbers to be the spectrum of a nonnegative matrix of size 5 and trace 0. This problem was solved by Laffey and Meehan [11] in 1999 with tools and ideas completely different from ours.

First we shall prove an auxiliary result. In what follows we denote the largest integer lower than or equal to the real number x by $[x]$.

Lemma 37 (Equitable separation of cycles). *If* $x^n + k_p x^{n-p} + \cdots + k_{2p}x^{n-2p} + \cdots + k_n$, $k_p \neq 0$, *is the characteristic polynomial of a weighted digraph* G, *then*

$$k_{2p} \leqslant \frac{1}{2}\left(1 - \frac{1}{\left[\frac{n}{p}\right]}\right)k_p^2. \tag{164}$$

Moreover, this inequality is optimum.

Proof. Observe that G has no cycle of length less than p. Firstly, let us see that the maximum value of k_{2p} is obtained when all the p-cycles of G are disjoint. Assume v_r is a vertex of G that is in more than one p-cycle and let $\{\widetilde{y}_i\}_{i=1}^t$ be the set of the p-cycles of G. Let $J_i = \{j : \widetilde{y}_i \cap \widetilde{y}_j = \emptyset\}$ and let $d_i = \sum_{j \in J_i} \Pi(\widetilde{y}_j)$, for $1 \leqslant i \leqslant t$, where if the index set J_i of some summatory is empty we will interpret it to be 0. Without loss of generality, we can assume that $\widetilde{y}_1, \ldots, \widetilde{y}_{t_r}$ are all the p-cycles of G in which v_r is present. Using the Coefficient Theorem we have

$$k_{2p} \leqslant \sum_{i=1}^{t_r} \Pi(\widetilde{y}_i)d_i + R_G, \tag{165}$$

where the summand R_G groups the contribution of the pairs of disjoint p-cycles which do not contain the vertex v_r. We can assume $d_1 = \max_{1 \leqslant i \leqslant t_r} d_i$. Let H be the weighted digraph obtained from G by deleting the arcs of the form $(*, v_r)$ of each p-cycle $\widetilde{y}_i$, $i = 2, \ldots, t_r$, and changing the weight w of the arc $(*, v_r)$ of the p-cycle $\widetilde{y}_1$ to

$$\frac{w}{\Pi(\widetilde{y}_1)} \sum_{i=1}^{t_r} \Pi(\widetilde{y}_i). \tag{166}$$

This digraph H has the same total weight of p-cycles as G, because the p-cycles that do not contain v_r have not been modified and the only p-cycle of H where v_r is present, $\widetilde{y}_1$, has weight $\sum_{i=1}^{t_r} \Pi(\widetilde{y}_i)$. Therefore

$$k_{2p} \leqslant d_1 \sum_{i=1}^{t_r} \Pi(\widetilde{y}_i) + R_G. \tag{167}$$

The iteration of this process allows us to assume that for a maximum k_{2p} the p-cycles of G are disjoint and so we have

$$k_{2p} \leqslant \sum_{1 \leqslant i < j \leqslant t} \Pi(\widetilde{y}_i)\Pi(\widetilde{y}_j). \tag{168}$$

Secondly, the maximum k_{2p} is attained when there exist at least two disjoint p-cycles and the weight of the p-cycles is equally distributed, *i.e.*, $\Pi(\widetilde{y}_i) = \frac{-k_p}{t}$, $i = 1, \ldots, t$. This is because if $\Pi(\widetilde{y}_i) < \Pi(\widetilde{y}_j)$ then the substitution of both weights for their mean will increase the value of k_{2p} preserving the value of k_p.

Finally, observe that

$$k_{2p} \leqslant \frac{k_p^2}{t^2}\binom{t}{2} = \frac{1}{2}\left(1 - \frac{1}{t}\right)k_p^2 \tag{169}$$

and then k_{2p} is maximum when t is maximum, which corresponds to $t = \left[\frac{n}{p}\right]$, *i.e.*, when the number of disjoint p-cycles is maximum. Only in this situation is the equality in (164) reached, which justifies its optimality. □

Remark 38. We can use the Newton identities (2) to express the inequality (164) in terms of the moments of the spectrum of a digraph G. It is enough to consider the cases $m = p$ and $m = 2p$ to obtain

$$p\left[\frac{n}{p}\right] s_{2p} \geqslant s_p^2, \quad \text{if } s_1 = \cdots = s_{p-1} = 0, \text{ for } 1 \leqslant p \leqslant \frac{n}{2}. \tag{170}$$

On the one hand, these expressions are a restricted refinement of the necessary condition of Johnson–Loewy–London $(s_k)^m \leqslant n^{m-1} s_{km}$, for $k, m = 1, 2, \ldots$. On the other hand, when $p = 2$ and n is odd we have $(n-1)s_4 \geqslant s_2^2$, so (170) is an extension of the necessary condition given by Laffey and Meehan [10] in 1998.

Theorem 39. *Let p and n be integers, such that $2 \leqslant p \leqslant n \leqslant 2p+1$. Let $P(x) = x^n + k_p x^{n-p} + \cdots + k_{n-1}x + k_n$. Then the following statements are equivalent*:

(i) *$P(x)$ is realizable*;
(ii) *the coefficients of $P(x)$ verify*:
  (a) $k_p, \ldots, k_{2p-1} \leqslant 0$;
  (b) $k_{2p} \leqslant \frac{k_p^2}{4}$;
  (c) $k_{2p+1} \leqslant \begin{cases} k_p k_{p+1} & \text{if } k_{2p} \leqslant 0, \\ k_{p+1}\left(\frac{k_p}{2} - \sqrt{\frac{k_p^2}{4} - k_{2p}}\right) & \text{if } k_{2p} > 0. \end{cases}$

Moreover, *when* (i) *and* (ii) *hold*, *$P(x)$ is EBL realizable*.

Proof. (i) implies (ii). Let G be a weighted digraph with characteristic polynomial $P(x)$. The Coefficient Theorem guarantees the condition (a), because of the absence of cycles of length lower than p. The condition (b) is deduced from the previous lemma. For the condition (c), in general, if $\{\widetilde{y}_i\}_{i=1}^t$ and $\{\widetilde{w}_j\}_{j=1}^r$ are the sets of cycles of lengths p and $p+1$ respectively, we have

$$k_{2p+1} \leqslant \sum_{\tilde{y}_i \cap \tilde{w}_j = \emptyset} \Pi(\widetilde{y}_i)\Pi(\widetilde{w}_j) \leqslant \left(\sum_{i=1}^t \Pi(\widetilde{y}_i)\right)\left(\sum_{j=1}^r \Pi(\widetilde{w}_j)\right) = (-k_p)(-k_{p+1}). \tag{171}$$

When $k_{2p} > 0$, there exists $m_0 \geqslant -\frac{k_p}{2}$ such that

$$k_{2p} = m_0(-k_p - m_0). \tag{172}$$

This situation corresponds to a digraph with two disjoint p-cycles with weights m_0 and $(-k_p - m_0)$ and without $2p$-cycles. This is the optimum situation because, if there are p-cycles with weights larger than m_0 then, with the following notations: $m = \max_{1 \leqslant i \leqslant t} \Pi(\widetilde{y}_i) = \Pi(\widetilde{y}_{i_m})$, for some $1 \leqslant i_m \leqslant t$, $J = \{j : \widetilde{y}_{i_m} \cap \widetilde{y}_j = \emptyset\}$, $Q = \{j \neq i_m : \widetilde{y}_{i_m} \cap \widetilde{y}_j \neq \emptyset\}$ and $Q_i = \{j : i < j \leqslant t : \widetilde{y}_i \cap \widetilde{y}_j = \emptyset\}$ for each $i \in Q$, we have

$$
\begin{aligned}
k_{2p} &\leqslant m\sum_{i\in J}\Pi(\widetilde{y}_i)+\sum_{i\in Q}\Pi(\widetilde{y}_i)\sum_{j\in Q_i}\Pi(\widetilde{y}_j)\\
&\leqslant m\sum_{i\in J}\Pi(\widetilde{y}_i)+\sum_{i\in Q}\Pi(\widetilde{y}_i)m=m(-k_p-m)\\
&< m_0(-k_p-m_0)=k_{2p},
\end{aligned}
\tag{173}
$$

where if the index set of some summatory is empty we will interpret it to be 0.

The maximum k_{2p+1} is obtained with the absence of $(2p+1)$-cycles and with a single $(p+1)$-cycle disjoint with the p-cycle of weight m_0. Thus $k_{2p+1}\leqslant m_0(-k_{p+1})$ with the largest m_0 verifying (172), *i.e.*,

$$
m_0=-\frac{k_p}{2}+\sqrt{\frac{k_p^2}{4}-k_{2p}}. \tag{174}
$$

(ii) implies (i). $P(x)$ is EBL realizable by the matrix $(a_{ij})_{i,j=1}^n$ where $a_{i,i+1}=1, i=1,\dots,n-1$ and otherwise $a_{ij}=0$, except for the following entries:

- If $n<2p$: $a_{i1}=-k_i, i=p,\dots,n$.
- If $n=2p$ and $k_{2p}\leqslant 0$: $a_{i1}=-k_i, i=p,\dots,n$.
- If $n=2p$ and $k_{2p}>0$: $a_{p1}=-\frac{k_p}{2}-\sqrt{\frac{k_p^2}{4}-k_{2p}}$; $a_{i1}=-k_i, i=p+1,\dots,n-1$; $a_{n,p+1}=-\frac{k_p}{2}+\sqrt{\frac{k_p^2}{4}-k_{2p}}$.
- If $n=2p+1$ and $k_{2p}\leqslant 0$: $a_{i1}=-k_i$, $i=p+1,\dots,2p$; $a_{2p+1,1}=k_pk_{p+1}-k_{2p+1}$; $a_{2p+1,p+2}=-k_p$.
- If $n=2p+1$ and $k_{2p}>0$: $a_{p1}=-\frac{k_p}{2}-\sqrt{\frac{k_p^2}{4}-k_{2p}}$; $a_{i1}=-k_i$, $i=p+1,\dots,2p-1$; $a_{2p+1,1}=k_{p+1}\left(\frac{k_p}{2}-\sqrt{\frac{k_p^2}{4}-k_{2p}}\right)-k_{2p+1}$; $a_{2p+1,p+2}=-\frac{k_p}{2}+\sqrt{\frac{k_p^2}{4}-k_{2p}}$. □

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