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A map of sufficient conditions for the real nonnegative inverse eigenvalue problem ☆

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Abstract

The real nonnegative inverse eigenvalue problem (RNIEP) is the problem of determining necessary and sufficient conditions for a list of real numbers Λ to be the spectrum of an entrywise nonnegative matrix. A number of sufficient conditions for the existence of such a matrix are known. In this paper, in order to construct a map of sufficient conditions, we compare these conditions and establish inclusion relations or independency relations between them.

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1. Introduction

The *nonnegative inverse eigenvalue problem* is the problem of characterizing all possible spectra $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of entrywise nonnegative matrices. This problem remains unsolved. Important advances towards a solution for an arbitrary n have been obtained by Loewy and London [8], Reams [14] and Laffey and Meehan [10,6].

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When Λ is a list of real numbers we have the *real nonnegative inverse eigenvalue problem* (hereafter RNIEP). This problem is only solved for $n \leqslant 4$ by Loewy and London [8]. A number of sufficient conditions have been obtained for the existence of a nonnegative matrix with prescribed real spectrum Λ. For a long time it was thought that the RNIEP was equivalent to the problem of characterizing the lists of real numbers which are the spectrum of nonnegative symmetric matrices. Johnson, Laffey and Loewy [4] in 1996 proved that both problems are different.

The first known sufficient conditions for the RNIEP were established for stochastic matrices [21,11,12,2]. As is well known, the real stochastic inverse eigenvalue problem is equivalent to the RNIEP and Kellogg [5] in 1971 gives the first condition for nonnegative matrices. Other conditions for the RNIEP in chronological order are in [15,3,20,1,22,16,19]. Only a few results are known about the relations between them. Our aim in this paper is to discuss those relations and to construct a map, which shows the inclusion relations and the independency relations between these sufficient conditions for the RNIEP.

Some of the sufficient conditions considered in this paper also hold for collections of complex numbers, see [22,7].

The paper is organized as follows: Section 2 contains the list of all sufficient conditions that we shall consider, in chronological order. Section 3 is devoted to establishing inclusion relations or independency relations between the distinct conditions.

2. Sufficient conditions for the RNIEP

In this paper we understand by a *list* a collection $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of real numbers with possible repetitions. By a *partition of a list* Λ we mean a family of sublists of Λ whose disjoint union is Λ. As is commonly accepted, we understand that a summatory is equal to zero when the index set of the summatory is empty.

We will say that a list Λ is *realizable* if it is the spectrum of an entrywise nonnegative matrix.

The RNIEP has an obvious solution when only nonnegative real numbers are considered, so the interest of the problem is when there is at least one negative number in the list.

An entrywise nonnegative matrix $A = (a_{ij})_{i,j=1}^n$ is said to have *constant row sums* if all its rows sum up to the same constant, say λ, *i.e.*

$$\sum_{j=1}^n a_{ij} = \lambda, \quad i = 1, \ldots, n.$$

The set of all entrywise nonnegative matrices with constant row sums equal to λ is denoted by $\mathscr{CS}_\lambda$.

In what follows we list most of the sufficient conditions for the RNIEP in chronological order. The first, and one of the most important results in this area was announced by Suleĭmanova [21] in 1949 and proved by Perfect [11] in 1953.

Theorem 2.1 (Suleĭmanova [21], 1949). *Let* $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ *satisfy*

$$\lambda_0 \geqslant |\lambda| \quad \text{for } \lambda \in \Lambda \text{ and } \lambda_0 + \sum_{\lambda_i < 0} \lambda_i \geqslant 0, \tag{1}$$

then Λ *is realizable in* $\mathscr{CS}_{\lambda_0}$.

We point out that this theorem has been extended to collections of complex numbers. In fact, Laffey and Šmigoc characterized when a collection of complex numbers $\{\rho, \lambda_2, \ldots, \lambda_n\}$, closed under complex conjugation, where $\rho > 0$ and $\mathrm{Re}(\lambda_j) \leqslant 0$, for $j = 2, \ldots, n$, is realizable (see [7, Theorem 3]).

The next result is a generalization of a condition given by Suleĭmanova [21, Theorem 3] and proved by Perfect [11, Theorem 3].

Theorem 2.2 (Suleĭmanova-Perfect [21,11], 1949–1953). *Let* $\Lambda = \{\lambda_0, \lambda_{01}, \ldots, \lambda_{0t_0}, \lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}, \ldots, \lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}\}$ *satisfy*

$$\lambda_0 \geqslant |\lambda| \quad \text{for } \lambda \in \Lambda \quad \text{and} \quad \lambda_j + \sum_{\lambda_{ji}<0} \lambda_{ji} \geqslant 0 \text{ for } j = 0, 1, \ldots, r, \tag{2}$$

then Λ *is realizable in* $\mathscr{CS}_{\lambda_0}$.

Definition 2.1. A set $\mathscr{K}$ of conditions is said to be a *realizability criterion* if any list of numbers Λ satisfying the conditions in $\mathscr{K}$ is realizable. In this case, we shall say that Λ is $\mathscr{K}$ *realizable*.

Definition 2.2. A list of numbers Λ is said to be *piecewise* $\mathscr{K}$ *realizable* if it can be partitioned as $\Lambda_1 \cup \cdots \cup \Lambda_t$ in such a way that Λ_i is $\mathscr{K}$ realizable for $i = 1, \ldots, t$.

In this paper $\mathscr{K}$, from the previous definitions, will be the surname of an author(s). For example, a list verifying Theorem 2.1 will be said to be Suleĭmanova realizable and if it verifies Theorem 2.2 it will be said to be Suleĭmanova-Perfect realizable and, in this case, also piecewise Suleĭmanova realizable.

Theorem 2.3 (Perfect 1 [11], 1953). *Let*

$$\Lambda = \{\lambda_0, \lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}, \ldots, \lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}, \delta\},$$

where

$$\lambda_0 \geqslant |\lambda| \quad \text{for } \lambda \in \Lambda, \ \sum_{\lambda\in\Lambda} \lambda \geqslant 0, \ \delta \leqslant 0,$$
$$\lambda_j \geqslant 0 \text{ and } \lambda_{ji} \leqslant 0 \quad \text{for } j = 1, \ldots, r \text{ and } i = 1, \ldots, t_j.$$

If

$$\lambda_j + \delta \leqslant 0 \text{ and } \lambda_j + \sum_{i=1}^{t_j} \lambda_{ji} \leqslant 0 \quad \text{for } j = 1, \ldots, r, \tag{3}$$

then Λ *is realizable in* $\mathscr{CS}_{\lambda_0}$.

Theorem 2.4 (Perfect 2 [12], 1955). *Let* $\{\lambda_0, \lambda_1, \ldots, \lambda_r\}$ *be realizable in* $\mathscr{CS}_{\lambda_0}$ *by a matrix with diagonal elements* $\omega_0, \omega_1, \ldots, \omega_r$ *and let* $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n\}$ *with* $-\lambda_0 \leqslant \lambda_i \leqslant 0$ *for* $i = r+1, \ldots, n$. *If there exists a partition* $\{\lambda_{01}, \ldots, \lambda_{0t_0}\} \cup \{\lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_{r1}, \ldots, \lambda_{rt_r}\}$ (*some or all of the lists may be empty*) *of* $\{\lambda_{r+1}, \ldots, \lambda_n\}$ *such that*

$$\omega_i + \sum_{j=1}^{t_i} \lambda_{ij} \geqslant 0 \quad \text{for } i = 0, 1, \ldots, r \tag{4}$$

then Λ *is realizable in* $\mathscr{CS}_{\lambda_0}$.

Although Perfect gives the previous theorem for stochastic matrices, the normal form of a stochastic matrix allows us to give the theorem for the nonnegative case. Note that originally the w_i's are diagonal elements of a stochastic matrix. When in the previous theorem the elements of the list $\{\lambda_0, \lambda_1, \ldots, \lambda_r\}$ are all nonnegative there always exists a realization of this list in $\mathcal{CS}_{\lambda_0}$. We will call this condition *Perfect* 2^+, *i.e.* Theorem 2.4 when $\lambda_i \geqslant 0$ for $i = 0, 1, \ldots, r$ (see [12, Theorem 3]).

All the previous conditions have proofs which are constructive, in the sense that they allow us to construct a realizing matrix.

In order to make use of Theorem 2.4, Perfect [12] gives sufficient conditions under which $\lambda_0, \lambda_1, \ldots, \lambda_r$ and $\omega_0, \omega_1, \ldots, \omega_r$ are the eigenvalues and the diagonal elements, respectively, of a matrix in $\mathcal{CS}_{\lambda_0}$. For $r = 1$ and $r = 2$ she gives necessary and sufficient conditions.

Lemma 2.1. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$, with $\lambda_1 \geqslant |\lambda|$ for $\lambda \in \Lambda$, realizable. The real numbers $\omega_1, \ldots, \omega_r$ are the diagonal elements of a matrix in $\mathcal{CS}_{\lambda_1}$ with spectrum Λ if*

(i) $0 \leqslant \omega_i \leqslant \lambda_1$, *for* $i = 1, \ldots, r$;
(ii) $\omega_1 + \cdots + \omega_r = \lambda_1 + \cdots + \lambda_r$;
(iii) $\omega_i \geqslant \lambda_i$ *and* $\omega_1 \geqslant \lambda_i$, *for* $i = 2, \ldots, r$.

Fiedler gives other sufficient conditions for the w_i's.

Lemma 2.2 (Fiedler [3], 1974). *Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, with $\lambda_1 \geqslant |\lambda_n|$, and $\omega_1 \geqslant \cdots \geqslant \omega_n \geqslant 0$ satisfy*

(i) $\sum_{i=1}^{s} \lambda_i \geqslant \sum_{i=1}^{s} \omega_i$ *for* $s = 1, \ldots, n-1$;
(ii) $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \omega_i$;
(iii) $\lambda_i \leqslant \omega_{i-1}$ *for* $i = 2, \ldots, n-1$.

Then there exists an $n \times n$ symmetric nonnegative matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and diagonal entries $\omega_1, \ldots, \omega_n$.

Theorem 2.5 (Ciarlet [2], 1968). *Let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ satisfy*

$$|\lambda_j| \leqslant \frac{\lambda_0}{n}, \quad j = 1, \ldots, n, \tag{5}$$

then Λ is realizable.

Theorem 2.6 (Kellogg [5], 1971). *Let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with $\lambda_0 \geqslant |\lambda|$ for $\lambda \in \Lambda$ and $\lambda_i \geqslant \lambda_{i+1}$ for $i = 0, \ldots, n-1$. Let M be the greatest index j $(0 \leqslant j \leqslant n)$ for which $\lambda_j \geqslant 0$ and $K = \{i \in \{1, \ldots, \lfloor n/2 \rfloor\} / \lambda_i \geqslant 0, \lambda_i + \lambda_{n+1-i} < 0\}$. If*

$$\lambda_0 + \sum_{i \in K, i<k} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-k} \geqslant 0 \quad \textit{for all } k \in K, \tag{6}$$

and

$$\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M} \lambda_j \geqslant 0, \tag{7}$$

then Λ is realizable.

Theorem 2.7 (Salzmann [15], 1972). *Let* $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ *with* $\lambda_i \geqslant \lambda_{i+1}$ *for* $i = 0, \ldots, n-1$. *If*

$$\sum_{0 \leqslant j \leqslant n} \lambda_j \geqslant 0, \tag{8}$$

and

$$\frac{\lambda_i + \lambda_{n-i}}{2} \leqslant \frac{1}{n+1} \sum_{0 \leqslant j \leqslant n} \lambda_j, \quad i = 1, \ldots, \lfloor n/2 \rfloor, \tag{9}$$

then Λ *is realizable by a diagonalizable nonnegative matrix.*

Theorem 2.8 (Fiedler [3], 1974). *Let* $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ *with* $\lambda_i \geqslant \lambda_{i+1}$ *for* $i = 0, \ldots, n-1$. *If*

$$\lambda_0 + \lambda_n + \sum_{\lambda \in \Lambda} \lambda \geqslant \frac{1}{2} \sum_{1 \leqslant i \leqslant n-1} |\lambda_i + \lambda_{n-i}|, \tag{10}$$

then Λ *is realizable by a symmetric nonnegative matrix.*

Soules in 1983 gives a constructive sufficient condition for symmetric realization. The inequalities that appear in this condition are obtained by imposing the diagonal elements of the matrix $P\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)P^t$ to be nonnegative, where P is an orthogonal matrix with a particular sign pattern (see [20, Lemma 2.1 and Lemma 2.2]).

Theorem 2.9 (Borobia [1], 1995). *Let* $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ *with* $\lambda_i \geqslant \lambda_{i+1}$ *for* $i = 0, \ldots, n-1$ *and let* M *be the greatest index* j $(0 \leqslant j \leqslant n)$ *for which* $\lambda_j \geqslant 0$. *If there exists a partition* $J_1 \cup \ldots \cup J_t$ *of* $\{\lambda_{M+1}, \ldots, \lambda_n\}$ *such that*

$$\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_M > \sum_{\lambda \in J_1} \lambda \geqslant \cdots \geqslant \sum_{\lambda \in J_t} \lambda \tag{11}$$

satisfies the Kellogg condition, then Λ *is realizable.*

Theorem 2.10 (Wuwen [22], 1997). *Let* $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ *be a realizable list with* $\lambda_1 \geqslant |\lambda|$ *for* $\lambda \in \Lambda$ *and let* ε_i *be real numbers for* $i = 2, \ldots, n$. *If* $\varepsilon_1 = \sum_{i=2}^{n} |\varepsilon_i|$, *then the list*

$$\{\lambda_1 + \varepsilon_1, \lambda_2 + \varepsilon_2, \ldots, \lambda_n + \varepsilon_n\}$$

is realizable.

We point out that Theorem 2.10 is a corollary of a result of Wuwen [22, Theorem 3.1] which holds for collections of complex numbers.

Theorem 2.11 (Soto 1 [16], 2003). *Let* $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ *with* $\lambda_i \geqslant \lambda_{i+1}$ *for* $i = 1, \ldots, n-1$. *Let* $S_k = \lambda_k + \lambda_{n-k+1}$, $k = 2, \ldots, \lfloor n/2 \rfloor$ *with* $S_{\frac{n+1}{2}} = \min\{\lambda_{\frac{n+1}{2}}, 0\}$ *for n odd. If*

$$\lambda_1 \geqslant -\lambda_n - \sum_{S_k < 0} S_k, \tag{12}$$

then Λ *is realizable in* $\mathcal{CS}_{\lambda_1}$.

In the context of Theorem 2.11 we define

$$T(\Lambda) = \lambda_1 + \lambda_n + \sum_{S_k<0} S_k.$$

In this way, (12) is equivalent to $T(\Lambda) \geqslant 0$. Observe that, if the list Λ is Soto 1 realizable then the new list

$$\left\{\lambda_1' = -\lambda_n - \sum_{S_k<0} S_k, \lambda_2, \ldots, \lambda_n\right\},$$

ordered decreasingly, is also Soto 1 realizable (if $\lambda_1' < \lambda_2$ the new inequality (12) is

$$\lambda_2 \geqslant -\lambda_n - \sum_{S_k<0} S_k$$

for the same $S_k < 0$).

Theorem 2.12 (Soto 2 [16], 2003). *Let Λ be a list that admits a partition*

$$\{\lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \ldots \cup \{\lambda_{r1}, \ldots, \lambda_{rt_r}\}$$

with $\lambda_{11} \geqslant |\lambda|$ for $\lambda \in \Lambda$, $\lambda_{ij} \geqslant \lambda_{i,j+1}$ and $\lambda_{i1} \geqslant 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, t_i$. For each list $\{\lambda_{i1}, \ldots, \lambda_{it_i}\}$ of the partition we define S_i and T_i as in Theorem 2.11, i.e.

$$\begin{aligned}
&S_{ij} = \lambda_{ij} + \lambda_{i,t_i-j+1} \quad \text{for } j = 2, \ldots, \lfloor t_i/2 \rfloor \\
&S_{i,(t_i+1)/2} = \min\{\lambda_{i,(t_i+1)/2}, 0\} \quad \text{if } t_i \text{ is odd for } i = 1, \ldots, r \\
&T_i = \lambda_{i1} + \lambda_{it_i} + \sum_{S_{ij}<0} S_{ij} \quad \text{for } i = 1, \ldots, r.
\end{aligned}$$

Let

$$L = \max\left\{-\lambda_{1t_1} - \sum_{S_{1j}<0} S_{1j},\ \max_{2\leqslant i\leqslant r}\{\lambda_{i1}\}\right\}. \tag{13}$$

If

$$\lambda_{11} \geqslant L - \sum_{T_i<0, 2\leqslant i\leqslant r} T_i, \tag{14}$$

then Λ is realizable in $\mathscr{CS}_{\lambda_{11}}$.

Theorem 2.13 (Soto–Rojo [19], 2006). *Let Λ be a list that admits a partition*

$$\{\lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \ldots \cup \{\lambda_{r1}, \ldots, \lambda_{rt_r}\}$$

with $\lambda_{11} \geqslant |\lambda|$ for $\lambda \in \Lambda$, $\lambda_{ij} \geqslant \lambda_{i,j+1}$ and $\lambda_{i1} \geqslant 0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, t_i$. Let $\omega_1, \ldots, \omega_r$ be nonnegative numbers such that there exists an $r \times r$ nonnegative matrix $B \in \mathscr{CS}_{\lambda_{11}}$ with eigenvalues $\lambda_{11}, \lambda_{21}, \ldots, \lambda_{r1}$ and diagonal entries $\omega_1, \ldots, \omega_r$. If the lists $\{\omega_i, \lambda_{i2}, \ldots, \lambda_{it_i}\}$ with $\omega_i \geqslant \lambda_{i2}$, for $i = 1, \ldots, r$, are realizable, then Λ is realizable in $\mathscr{CS}_{\lambda_{11}}$.

The sufficient conditions of Salzmann, Soto 1, Soto 2 and Soto–Rojo have constructive proofs, which allow us to compute an explicit realizing matrix.

3. Inclusion relations

In this section we compare the realizability criteria for the RNIEP. In what follows, we will understand by *Fiedler* the sufficient condition for the RNIEP given at Theorem 2.8.

Theorem 3.1

1. *Ciarlet implies Suleĭmanova and the inclusion is strict.*
2. *Ciarlet, Suleĭmanova and Suleĭmanova-Perfect are independent of Salzmann.*
3. *Suleĭmanova implies Fiedler and the inclusion is strict.*
4. *Suleĭmanova-Perfect is independent of Fiedler.*

Proof. 1. Let $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ verify the Ciarlet condition: $|\lambda_j| \leqslant \frac{\lambda_0}{n}$ for $j = 1, \dots, n$. Then $\lambda_0 \geqslant n|\lambda_j| \geqslant |\lambda_j|$ for $j = 1, \dots, n$ and

$$\lambda_0 + \sum_{\lambda_j<0} \lambda_j \geqslant \lambda_0 + \sum_{\lambda_j<0} \frac{-\lambda_0}{n} \geqslant 0,$$

so Λ verifies the Suleĭmanova condition. $\Lambda = \{2, 0, -2\}$ shows the inclusion is strict.

2. The list $\{2, 1, -1\}$ verifies Ciarlet and Suleĭmanova but not Salzmann and $\{3, 1, -2, -2\}$ verifies Salzmann but not the Ciarlet nor Suleĭmanova-Perfect conditions.

3. Let $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ verify the Suleĭmanova condition. We can assume $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n$. We will prove the result for n even and $\lambda_{n/2} < 0$ because for $\lambda_{n/2} \geqslant 0$ and n odd the proofs are similar. For this situation we have

$$\begin{aligned} &2(\lambda_0 + \lambda_n) + \sum_{j=1}^{n-1} \lambda_j - \frac{1}{2} \sum_{j=1}^{n-1} |\lambda_j + \lambda_{n-j}| \\ &\quad = 2(\lambda_0 + \lambda_n) + \sum_{j=1}^{\frac{n}{2}-1} \left(\lambda_j + \lambda_{n-j} - |\lambda_j + \lambda_{n-j}|\right) + \lambda_{\frac{n}{2}} - |\lambda_{\frac{n}{2}}| \\ &\quad = 2(\lambda_0 + \lambda_n) + 2 \sum_{\substack{j=1 \\ \lambda_j+\lambda_{n-j}<0}}^{\frac{n}{2}-1} (\lambda_j + \lambda_{n-j}) + 2\lambda_{\frac{n}{2}} \geqslant 0 \end{aligned}$$

where the last inequality is verified because of the Suleĭmanova condition. The list $\{1, 1, -1, -1\}$ shows the inclusion is strict.

4. The list $\{3, 2, -1, -1, -3\}$ verifies Suleĭmanova-Perfect but not Fiedler and $\{4, 2, 1, -3, -3\}$ verifies Fiedler but not Suleĭmanova-Perfect. □

Fiedler proves in [3] that his condition includes the Salzmann condition. Moreover, since Suleĭmanova-Perfect is a piecewise Suleĭmanova condition and Suleĭmanova implies Fiedler, then Suleĭmanova-Perfect implies the piecewise Fiedler condition. As a conclusion, we observe that the piecewise Fiedler realizability criterion contains all realizability criteria in Theorem 3.1.

Theorem 3.2. *Soto* 1 *is equivalent to Fiedler.*

Proof. Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, S_k and $S_{\frac{n+1}{2}}$ be as in Soto 1. If Λ verifies Fiedler then

$$\lambda_1 + \lambda_n + \sum_{k=1}^{n} \lambda_k \geqslant \frac{1}{2} \sum_{k=2}^{n-1} |\lambda_k + \lambda_{n-k+1}|.$$

This inequality can be written as

$$2(\lambda_1 + \lambda_n) + \sum_{k=2}^{n-1} \lambda_k \geqslant \begin{cases} \sum_{k=2}^{n/2} |\lambda_k + \lambda_{n-k+1}| & \text{for } n \text{ even,} \\ \sum_{k=2}^{\lfloor n/2 \rfloor} |\lambda_k + \lambda_{n-k+1}| + \left|\lambda_{\frac{n+1}{2}}\right| & \text{for } n \text{ odd} \end{cases}$$

and in both cases it is equivalent to

$$2(\lambda_1 + \lambda_n) \geqslant -2 \sum_{S_k < 0} S_k$$

which is Soto 1. □

The Soto result is constructive while the Fiedler result is not.

Fiedler proves in [3] that his condition implies the Kellogg condition and that the inclusion is strict. Fiedler also proves that the Kellogg condition guarantees symmetric realization. It is well known that Kellogg implies Borobia.

Theorem 3.3

1. *Suleǐmanova-Perfect and Kellogg are independent.*
2. *Suleǐmanova-Perfect implies Borobia and the inclusion is strict.*

Proof. 1. The list $\{3, 1, -2, -2\}$ verifies Kellogg and not Suleǐmanova-Perfect. The list $\{3, 3, -1, -1, -2, -2\}$ verifies Suleǐmanova-Perfect and not Kellogg.

2. Let $\Lambda = \{\lambda_0, \lambda_{01}, \ldots, \lambda_{0t_0}, \lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}, \ldots, \lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}\}$ verify Suleǐmanova-Perfect: $\lambda_0 \geqslant |\lambda|$ for $\lambda \in \Lambda$ and $\lambda_j + \sum_{\lambda_{ji}<0} \lambda_{ji} \geqslant 0$ for $j = 0, 1, \ldots, r$. We can assume $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_r$. We can also assume that for $\lambda_{ij} \geqslant 0$ we have $\lambda_r \geqslant \lambda_{ij}$: if there exist indexes $0 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant t_i$ with $\lambda_{ij} > \lambda_r$ we can exchange them because

$$\lambda_{ij} + \sum_{\lambda_{rk}<0} \lambda_{rk} > \lambda_r + \sum_{\lambda_{rk}<0} \lambda_{rk} \geqslant 0.$$

Let $\lambda_{r+1} \geqslant \cdots \geqslant \lambda_M$ be the ordered list $\{\lambda \in \Lambda/\lambda \geqslant 0, \lambda \neq \lambda_i \ i = 0, 1, \ldots, r\}$.
If Λ has no negative elements the result is clear; otherwise, we can assume $\{\lambda_{ij} < 0, j = 1, \ldots, t_i\} \neq \emptyset$ for $0 \leqslant i \leqslant r$. Let us define $\mu_i = \sum_{\lambda_{ij}<0} \lambda_{ij}$ for $i = 0, 1, \ldots, r$. We can also assume that $\mu_r \geqslant \cdots \geqslant \mu_1 \geqslant \mu_0$: if there exist indexes $i < j$ with $\mu_i > \mu_j$ we can exchange them because

$$\mu_i > \mu_j \Longrightarrow \lambda_j + \mu_i > \lambda_j + \mu_j \geqslant 0,$$
$$\lambda_i \geqslant \lambda_j \Longrightarrow \lambda_i + \mu_j \geqslant \lambda_j + \mu_j \geqslant 0.$$

We shall now prove that

$$\hat{\Lambda} = \{\lambda_0 \geqslant \cdots \geqslant \lambda_r \geqslant \lambda_{r+1} \geqslant \cdots \geqslant \lambda_M > \mu_r = \lambda_{M+1} \geqslant \cdots \geqslant \mu_0 = \lambda_{M+r+1}\}$$

verifies Kellogg. Let $K = \{i \in \{1, \ldots, r+1\}/\lambda_i + \mu_{i-1} < 0\} = \{i_1 < \cdots < i_q\}$. Condition (6) is verified because, for all $p \leqslant q$, we have

$$\lambda_0 + \sum_{j=1}^{p-1}(\lambda_{i_j} + \mu_{i_j-1}) + \mu_{i_p-1} = (\lambda_0 + \mu_{i_1-1}) + \sum_{j=1}^{p-2}(\lambda_{i_j} + \mu_{i_{j+1}-1}) + (\lambda_{i_{p-1}} + \mu_{i_p-1})$$
$$\geqslant (\lambda_0 + \mu_0) + \sum_{j=1}^{p-1}(\lambda_{i_j} + \mu_{i_j}) \geqslant 0.$$

For condition (7) we observe that $\sum_{j=M+1}^{r+1}\lambda_j = \mu_r$ if $M = r$ and $\sum_{j=M+1}^{r+1}\lambda_j = 0$ in other cases. If $r+1 \notin K$, then

$$\lambda_0 + \sum_{j=1}^{q}(\lambda_{i_j} + \mu_{i_j-1}) + \sum_{j=M+1}^{r+1}\lambda_j$$
$$\geqslant \lambda_0 + \sum_{j=1}^{q}(\lambda_{i_j} + \mu_{i_j-1}) + \mu_r$$
$$= (\lambda_0 + \mu_{i_1-1}) + \sum_{j=1}^{q-1}(\lambda_{i_j} + \mu_{i_{j+1}-1}) + (\lambda_{i_q} + \mu_r)$$
$$\geqslant (\lambda_0 + \mu_0) + \sum_{j=1}^{q-1}(\lambda_{i_j} + \mu_{i_j}) + (\lambda_r + \mu_r) \geqslant 0.$$

If $r+1 \in K$, then the former inequalities hold replacing μ_r by 0.

The previous example $\{3, 1, -2, -2\}$ shows the inclusion is strict. □

Theorem 3.4

1. *Ciarlet, Suleĭmanova, Suleĭmanova-Perfect, Salzmann, Soto* 1 *and Kellogg are independent of Perfect* 1.
2. *Perfect* 1 *with all the lists of nonpositive numbers*, $\{\lambda_{i1}, \ldots, \lambda_{it_i}\}$, *with one element implies Kellogg*.
3. *If* $\Lambda = \{\lambda_0, \lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}, \ldots, \lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}, \delta\}$ *verifies Perfect* 1 *then* $\widetilde{\Lambda} = \{\lambda_0, \lambda_1, \sum_{j=1}^{t_1}\lambda_{1j}, \ldots, \lambda_r, \sum_{j=1}^{t_r}\lambda_{rj}, \delta\}$ *too*.
4. *Perfect* 1 *implies Borobia and the inclusion is strict*.

Proof. 1. The list $\{5, 4, 3, -2, -2, -2, -2, -4\}$ verifies Perfect 1 but not any of the other conditions. The list $\{3, 1, -1\}$ verifies Ciarlet (so Suleĭmanova, Suleĭmanova-Perfect, Soto 1 (Fiedler) and Kellogg too) and Salzmann but not Perfect 1.

2. Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_{11}, \ldots, \lambda_r, \lambda_{r1}, \delta\}$ verify Perfect 1: $\lambda_0 \geqslant |\lambda|$ for $\lambda \in \Lambda$, $\sum_{\lambda\in\Lambda}\lambda \geqslant 0$, $\lambda_j \geqslant 0$ and $\lambda_{j1} \leqslant 0$ for $j = 1, \ldots, r$ and $\delta \leqslant 0$, $\lambda_j + \delta \leqslant 0$ and $\lambda_j + \lambda_{j1} \leqslant 0$ for $j = 1, \ldots, r$. We can assume $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_r$. We can also assume that $\delta = \min\{\delta, \lambda_{j1}, j = 1, \ldots, r\}$: if there exists an index i with $\delta > \lambda_{i1}$ we can exchange them because

$$\lambda_j + \lambda_{i1} < \lambda_j + \delta \leqslant 0 \quad j = 1, \ldots, r \text{ and } \lambda_i + \delta \leqslant 0.$$

We can also assume that $\lambda_{r1} \geqslant \cdots \geqslant \lambda_{11}$: if there exists an index $i < j$ with $\lambda_{i1} > \lambda_{j1}$ we can exchange them because

$$\lambda_{j1} < \lambda_{i1} \Longrightarrow \lambda_i + \lambda_{j1} < \lambda_i + \lambda_{i1} \leqslant 0,$$
$$\lambda_i \geqslant \lambda_j \Longrightarrow \lambda_j + \lambda_{i1} \leqslant \lambda_i + \lambda_{i1} \leqslant 0.$$

Now let us see that the list

$$\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant \lambda_{r1} = \lambda_{r+1} \geqslant \cdots \geqslant \lambda_{11} = \lambda_{2r} \geqslant \delta = \lambda_{2r+1}$$

satisfies Kellogg. Let

$$K = \left\{ i \in \left\{ 1, \ldots, \left\lfloor \frac{2r+1}{2} \right\rfloor \right\} \Big/ \lambda_i \geqslant 0, \lambda_i + \lambda_{2r+2-i} < 0 \right\} \subseteq \{1, \ldots, r\}$$

and

$$M = \max\{i/\lambda_i \geqslant 0\} = \begin{cases} r & \text{if } \lambda_{r1} < 0 \\ r+s, s \geqslant 1 & \text{if } \lambda_{r+s} = 0 \text{ and } \lambda_{r+s+1} < 0. \end{cases}$$

Note that if $i \in \{1, \ldots, r\}$ then

$$\begin{cases} \lambda_i + \lambda_{2r+2-i} < 0 \text{ and } i \in K \\ \text{or} \\ \lambda_i + \lambda_{2r+2-i} = 0 \text{ and } i \notin K, \end{cases}$$

so for $k \in K$ we have

$$\begin{aligned} &\lambda_0 + \sum_{\substack{i \in K \\ i < k}} (\lambda_i + \lambda_{2r+2-i}) + \lambda_{2r+2-k} \\ &\quad = \lambda_0 + \sum_{i=1}^{k-1} (\lambda_i + \lambda_{2r+2-i}) + \lambda_{2r+2-k} \\ &\quad = \lambda_0 + \delta + \sum_{i=1}^{k-1} (\lambda_i + \lambda_{i1}) \\ &\quad \geqslant \lambda_0 + \delta + \sum_{i=1}^{k-1} (\lambda_i + \lambda_{i1}) + \sum_{i=k}^{r} (\lambda_i + \lambda_{i1}) = \sum_{\lambda \in \Lambda} \lambda \geqslant 0 \end{aligned}$$

which proves (6). Now let us see that (7) holds:

$$\begin{aligned} &\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{2r+2-i}) + \sum_{j=M+1}^{2r+1-M} \lambda_j \\ &\quad = \lambda_0 + \sum_{i=1}^{r} (\lambda_i + \lambda_{2r+2-i}) + \sum_{j=M+1}^{2r+1-M} \lambda_j = \sum_{\lambda \in \Lambda} \lambda \geqslant 0. \end{aligned}$$

3. It is enough to prove $\lambda_0 + \sum_{j=1}^{t_i} \lambda_{ij} \geqslant 0$ for $i = 1, \ldots, r$:

$$\lambda_0 + \sum_{j=1}^{t_i} \lambda_{ij} \geqslant \lambda_0 + \sum_{j=1}^{t_i} \lambda_{ij} + \lambda_i + \delta + \sum_{\substack{k=1 \\ k \neq i}}^{r} \left(\lambda_k + \sum_{j=1}^{t_k} \lambda_{kj} \right) = \sum_{\lambda \in \Lambda} \lambda \geqslant 0.$$

The inequality is given by the Perfect 1 conditions applied to Λ: $\lambda_i + \delta \leqslant 0$ and $\lambda_k + \sum_{j=1}^{t_k} \lambda_{kj} \leqslant 0$.

4. The inclusion is clear from Claims 2 and 3 of this theorem and because Kellogg implies Borobia. The list $\{3, 3, -1, -1, -2, -2\}$ verifies Borobia but not Perfect 1. □

Radwan [13] in 1996 proved that the Borobia condition guarantees symmetric realization.

As a conclusion from the comparison of the previous realizability criteria, we observe that the Borobia realizability criterion contains all of them. The next diagram describes this conclusion

$$\begin{array}{ccccccccc}
\text{Ciarlet} & \Longrightarrow & \text{Suleĭmanova} & \Longrightarrow & \text{Suleĭmanova-Perfect} & & & & \\
& & \searrow & & & & \searrow & & \\
& & & \text{Fiedler} \Longleftrightarrow \text{Soto } 1 \Longrightarrow & \text{Kellogg} & \Longrightarrow & & \text{Borobia.} & \\
& & \nearrow & & & & \nearrow & & \\
& \text{Salzman} & & & & \text{Perfect } 1 & & &
\end{array}$$

Theorem 3.5

1. *Piecewise Soto* 1 *implies Soto* 2 *and the inclusion is strict.*
2. *Suleĭmanova-Perfect implies Soto* 2 *and the inclusion is strict.*
3. *Perfect* 1 *implies Soto* 2 *and the inclusion is strict.*
4. *Kellogg and Borobia are independent of Soto* 2.

Proof. 1. The inclusion is clear. The list $\{8, 3, -5, -5\} \cup \{6, 3, -5, -5\}$ shows it is strict.

2. We know that Suleĭmanova implies Fiedler (Soto 1), Theorem 3.1 3., so Suleĭmanova-Perfect implies piecewise Soto 1 and also, because of the previous result, Soto 2. The list $\{3, 1, -2, -2\}$ shows the inclusion is strict.

3. Let

$$\Lambda = \{\lambda_0, \lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}, \ldots, \lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}, \delta\}$$

verify Perfect 1: $\lambda_0 \geqslant |\lambda|$ for $\lambda \in \Lambda$, $\sum_{\lambda\in\Lambda} \lambda \geqslant 0$, $\delta \leqslant 0$, $\lambda_j \geqslant 0$ and $\lambda_{ji} \leqslant 0$ for $j = 1, \ldots, r$ and $i = 1, \ldots, t_j$, $\lambda_j + \delta \leqslant 0$ and $\lambda_j + \sum_{i=1}^{t_j} \lambda_{ji} \leqslant 0$ for $j = 1, \ldots, r$. Let us see that the partition of Λ

$$\Lambda = \{\lambda_0, \delta\} \cup \{\lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \ldots \cup \{\lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}\}$$

verifies Soto 2. Using the notation of this criterion, Theorem 2.12, we have $T_j = \lambda_j + \sum_{i=1}^{t_j} \lambda_{ji} \leqslant 0$ for $j = 1, \ldots, r$ and $L = -\delta$. Finally

$$\lambda_0 - L + \sum_{T_j<0, 1\leqslant j\leqslant r} T_j = \lambda_0 + \delta + \sum_{j=1}^{r} T_j = \sum_{\lambda\in\Lambda} \lambda \geqslant 0,$$

proves the result.

4. The list $\{3, 3, 1, 1, -2, -2, -2, -2\}$ verifies Soto 2, with the partition $\{3, 1, -2, -2\} \cup \{3, 1, -2, -2\}$, but not Kellogg nor Borobia. The list $\{9, 7, 4, -3, -3, -6, -8\}$ verifies Kellogg but not Soto 2. □

Remark 3.1. In [18] it was shown that the Kellogg and Borobia conditions imply, in a "certain sense", the Soto 2 condition. In [17] it was shown that the Soto 2 condition guarantees symmetric realization.

Lemma 3.1. *Let $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n\}$ verify Perfect 2 (respectively Perfect 2^+) and let $q \in \{r+1, \ldots, n\}$. If $\lambda_q = \sum_{1 \leqslant j \leqslant p} \mu_j$ with $\mu_j \leqslant 0$, then*

$$\{\lambda_0, \lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_{q-1}, \mu_1, \ldots, \mu_p, \lambda_{q+1}, \ldots, \lambda_n\}$$

also verifies Perfect 2 (respectively Perfect 2^+).

Proof. The result is clear because by replacing λ_q by $\mu_1, \ldots, \mu_p$ does not change the existence of the matrix in $\mathscr{CS}_{\lambda_0}$, with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_r$ and diagonal elements $\omega_0, \omega_1, \ldots, \omega_r$, nor the realizability of the new list. □

Perfect [12] proved that Perfect 2^+ includes Suleǐmanova-Perfect and Perfect 1.

Theorem 3.6. *Borobia implies Perfect 2^+ and the inclusion is strict.*

Proof. From Lemma 3.1 it is enough to prove the result for $\Lambda = \{\lambda_0, \ldots, \lambda_n\}$ verifying Kellogg: $\lambda_0 \geqslant \cdots \geqslant \lambda_n$, $\lambda_0 \geqslant |\lambda_n|$, $K = \{i \in \{1, \ldots, \lfloor n/2 \rfloor\} / \lambda_i \geqslant 0, \lambda_i + \lambda_{n+1-i} < 0\}$, $M = \max\{j \in \{0, \ldots, n\} / \lambda_j \geqslant 0\}$ and the conditions

$$\text{K1:}\ \lambda_0 + \sum_{i \in K, i<k} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-k} \geqslant 0 \quad \text{for all } k \in K,$$

$$\text{K2:}\ \lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M} \lambda_j \geqslant 0.$$

Let us see that $\{\lambda_{M+1}, \ldots, \lambda_n\}$ can be partitioned as $\{\lambda_{01}, \ldots, \lambda_{0t_0}\} \cup \{\lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_{M1}, \ldots, \lambda_{Mt_M}\}$ in such a way that there exists a list $W = \{\omega_0, \ldots, \omega_M\}$ formed by the diagonal elements of a realization of $\{\lambda_0, \ldots, \lambda_M\}$ such that

$$\omega_i + \sum_{j=1}^{t_i} \lambda_{ij} \geqslant 0 \quad \text{for } i = 0, 1, \ldots, M. \tag{15}$$

Let us consider the partition

$$\{\lambda_{M+1}, \ldots, \lambda_{n-M}\}, \{\lambda_n\}, \{\lambda_{n-1}\}, \ldots, \{\lambda_{n-M+1}\}$$

and the list

$$W = \left\{ \lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{n+1-i}),\ \max\{\lambda_i, -\lambda_{n+1-i}\},\ i = 1, \ldots, M \right\}.$$

It is clear that the lists of the partition and the elements of W can be coupled (in fact, in the order they are written) to verify (15).

Let us see that W can be the list of diagonal elements of a realization of $\{\lambda_0, \ldots, \lambda_M\}$ showing that they verify the sufficient conditions due to Perfect, Lemma 2.1, or to Fiedler, Lemma 2.2. Condition (ii) in both lemmas is the same and is satisfied:

$$\begin{aligned} \sum_{\omega \in W} \omega &= \lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{n+1-i}) + \sum_{i=1}^{M} \max\{\lambda_i, -\lambda_{n+1-i}\} \\ &= \lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{n+1-i}) + \sum_{i \notin K} \lambda_i - \sum_{i \in K} \lambda_{n+1-i} = \sum_{i=0}^{M} \lambda_i. \end{aligned}$$

Conditions (i) and (iii) in both lemmas are different and depend on the indexing of W or the order of its elements.

We have the following cases:

(a) $\lambda_0+\sum_{i\in K}(\lambda_i+\lambda_{n+1-i})\geqslant\lambda_1$. It can be seen that by indexing W as

$$\omega_0=\lambda_0+\sum_{i\in K}(\lambda_i+\lambda_{n+1-i}),\ \omega_i=\max\{\lambda_i,-\lambda_{n+1-i}\}\quad\text{for } i=1,\dots,M,$$

conditions (i) and (iii) of Lemma 2.1 are satisfied.

(b) $\lambda_0+\sum_{i\in K}(\lambda_i+\lambda_{n+1-i})<\lambda_1$. Let us index W in decreasing order and see that Fiedler's conditions from Lemma 2.2 are satisfied. We observe that

$$\max\{\lambda_i,-\lambda_{n+1-i}\}\geqslant\max\{\lambda_{i+1},-\lambda_{n-i}\}\quad\text{for } i=1,\dots,M-1,$$

so the order of the ω_i's only depends on the position of $\lambda_0+\sum_{i\in K}(\lambda_i+\lambda_{n+1-i})$ among them. Assume

$$\begin{aligned}\omega_0&=\max\{\lambda_1,-\lambda_n\}\geqslant\cdots\geqslant\omega_{p-1}=\max\{\lambda_p,-\lambda_{n+1-p}\}\\&\geqslant\omega_p=\lambda_0+\sum_{i\in K}(\lambda_i+\lambda_{n+1-i})\geqslant\omega_{p+1}\\&\geqslant\cdots\geqslant\max\{\lambda_M,-\lambda_{n+1-M}\}\end{aligned}$$

for an index $p\in\{1,\dots,M\}$.

Let us prove condition (i) of Lemma 2.2 by induction. Clearly $\omega_0\leqslant\lambda_0$. Let us see that $\sum_{0\leqslant i\leqslant m-1}\omega_i\leqslant\sum_{0\leqslant i\leqslant m-1}\lambda_i$, for an index m with $1\leqslant m\leqslant M-1$, implies $\sum_{0\leqslant i\leqslant m}\omega_i\leqslant\sum_{0\leqslant i\leqslant m}\lambda_i$. It can happen that:

(b1) $m<p$. If $m+1\notin K$, then $\omega_m=\max\{\lambda_{m+1},-\lambda_{n-m}\}=\lambda_{m+1}\leqslant\lambda_m$ and

$$\sum_{i=0}^{m}\omega_i\leqslant\sum_{i=0}^{m-1}\lambda_i+\omega_m\leqslant\sum_{i=0}^{m}\lambda_i.$$

If $m+1\in K$, the Kellogg condition K1 for $k=m+1$ gives

$$-\lambda_{n-m}\leqslant\lambda_0+\sum_{i\in K,i\leqslant m}(\lambda_i+\lambda_{n+1-i}).$$

Therefore

$$\begin{aligned}\sum_{i=0}^{m}\omega_i&=\sum_{i=0}^{m-1}\max\{\lambda_{i+1},-\lambda_{n-i}\}+\omega_m\\&=\sum_{i\notin K,i\leqslant m}\lambda_i-\sum_{i\in K,i\leqslant m}\lambda_{n+1-i}-\lambda_{n-m}\\&\leqslant\sum_{i\notin K,i\leqslant m}\lambda_i-\sum_{i\in K,i\leqslant m}\lambda_{n+1-i}+\lambda_0+\sum_{i\in K,i\leqslant m}(\lambda_i+\lambda_{n+1-i})\\&=\sum_{i=0}^{m}\lambda_i.\end{aligned}$$

(b2) $m\geqslant p$. In this case we have

$$\begin{aligned}\sum_{i=0}^{m}\omega_i&=\sum_{i=0}^{p-1}\max\{\lambda_{i+1},-\lambda_{n-i}\}+\omega_p+\sum_{i=p+1}^{m}\max\{\lambda_i,-\lambda_{n+1-i}\}\\&=\sum_{i=1}^{m}\max\{\lambda_i,-\lambda_{n+1-i}\}+\lambda_0+\sum_{i\in K}(\lambda_i+\lambda_{n+1-i})\end{aligned}$$

$$= \lambda_0 + \sum_{i \notin K, i \leqslant m} \lambda_i - \sum_{i \in K, i \leqslant m} \lambda_{n+1-i} + \sum_{i \in K, i \leqslant m} (\lambda_i + \lambda_{n+1-i}) + \sum_{i \in K, i > m} (\lambda_i + \lambda_{n+1-i}) \leqslant \sum_{i=0}^{m} \lambda_i.$$

Finally condition (iii) of Lemma 2.2 can be easily verified.

The list {6, 1, 1, −4, −4} shows that the inclusion is strict. In fact, the matrix

$$\begin{pmatrix} 4 & 0 & 2 \\ 3/2 & 4 & 1/2 \\ 0 & 6 & 0 \end{pmatrix}$$

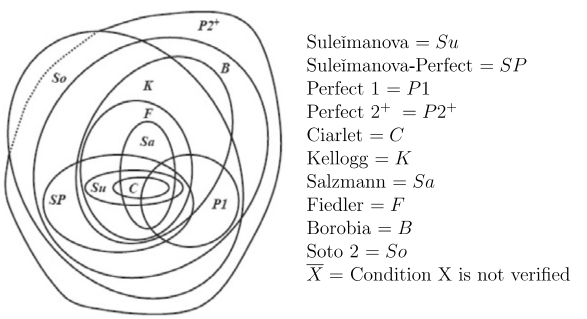
has spectrum {6, 1, 1}, diagonal entries 4, 4, 0 and the sum of the elements of each one of the lists {4, −4}, {4, −4}, {0} is nonnegative. □

The list {6, 1, 1, −4, −4} shows that Perfect 2^+ does not imply Soto 2 and we do not know if Soto 2 implies (or not) Perfect 2^+.

With the exception of the Soto 2 realizability criterion, we observe that the Perfect 2^+ realizability criterion contains all the criteria compared in this section. The Soto–Rojo condition extends the Perfect 2^+ realizability criterion, allowing a more general partition accomplished on a negative portion of the list Λ. In [19] it is shown that Soto 2 implies Soto–Rojo and the inclusion is strict. Thus, Soto–Rojo contains all realizability criteria, which we have compared in this work, but we do not know if the inclusion of Perfect 2^+ in Soto–Rojo is strict.

Finally, we observe that the Wuwen realizability criterion is an atypical criterion, in the sense that it needs a realizable list as a hypothesis.

Next we show a map with all the relations between the sufficient conditions studied and we give a collection of examples to explain them. Part of this map appears in [9].



$\overline{P1} \cap C$: {3, 1, −1},

$P1 \cap C$: {3, 1, −1, −1},

$\overline{P1} \cap C \cap \overline{Sa}$: {2, 1, −1},

$\overline{P1} \cap Su \cap \overline{C} \cap Sa$: {2, 0, −2},

$\overline{P1} \cap Su \cap \overline{C} \cap \overline{Sa}$: {1, 1, −1},

$P1 \cap Su \cap \overline{C} \cap Sa$: {5, 2, −2, −3},

$P1 \cap Su \cap \overline{C} \cap \overline{Sa}$: {5, 2, −1, −1, −3},

$\overline{Su} \cap SP \cap P1 \cap Sa : \{1, 1, -1, -1\}$,
$\overline{Su} \cap SP \cap P1 \cap \overline{Sa} \cap F : \{4, 3, -1, -2, -3\}$,
$\overline{Su} \cap SP \cap P1 \cap \overline{F} \cap K : \{3, 2, -1, -1, -3\}$,
$\overline{Su} \cap SP \cap P1 \cap \overline{K} : \{3, 2, 2, -1, -1, -1, -1, -1, -2\}$,
$\overline{Su} \cap SP \cap \overline{P1} \cap Sa : \{2, 2, 0, -1, -2\}$,
$\overline{Su} \cap SP \cap \overline{P1} \cap \overline{Sa} \cap F : \{5, 4, -2, -4\}$,
$\overline{Su} \cap SP \cap \overline{P1} \cap \overline{F} \cap K : \{6, 5, 1, -3, -3, -5\}$,
$\overline{Su} \cap SP \cap \overline{P1} \cap \overline{K} : \{3, 3, -1, -1, -2, -2\}$,
$\overline{SP} \cap P1 \cap Sa : \{3, 1, -2, -2\}$,
$\overline{SP} \cap P1 \cap \overline{Sa} \cap F : \{5, 2, 2, -1, -1, -3, -3\}$,
$\overline{SP} \cap P1 \cap \overline{F} \cap K : \{6, 4, 0, -2, -3, -5\}$,
$\overline{SP} \cap P1 \cap \overline{K} : \{5, 4, 3, -2, -2, -2, -2, -4\}$,
$\overline{SP} \cap \overline{P1} \cap Sa : \{14, 6, 1, -7, -8\}$,
$\overline{SP} \cap \overline{P1} \cap \overline{Sa} \cap F : \{4, 2, 1, -3, -3\}$,
$\overline{SP} \cap \overline{P1} \cap \overline{F} \cap K \cap So : \{6, 4, 1, -3, -3, -5\}$,
$\overline{SP} \cap \overline{P1} \cap \overline{F} \cap K \cap \overline{So} : \{9, 7, 4, -3, -3, -6, -8\}$,
$\overline{SP} \cap \overline{P1} \cap \overline{K} \cap B \cap So : \{5, 3, -2, -2, -2, -2\}$,
$\overline{SP} \cap \overline{P1} \cap \overline{K} \cap B \cap \overline{So} : \{7, 7, 5, -3, -3, -3, -3, -6\}$,
$\overline{B} \cap So \cap P2^+ : \{3, 3, 1, 1, -2, -2, -2, -2\}$,
$\overline{B} \cap So \cap \overline{P2^+}$:?,
$\overline{B} \cap \overline{So} \cap P2^+ : \{6, 1, 1, -4, -4\}$.

Finally the list $\{7, 5, -4, -4, -4\}$ is realizable but does not verify the Soto nor Perfect 2^+ conditions. This list verifies the necessary and sufficient conditions given in [6].

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