JTN 2015
Universidad de Valladolid
June 29-July 3, 2015

Introduction to Logarithmic Classes

J-I. Villanueva[1]

[1] *Université de Bordeaux, France, jovillan@math.u-bordeaux1.fr*

In this talk I will present the advances on my research of my first year of PhD at the University of Bordeaux under the advise of the professor Jean-François Jaulent.

Let K be a number field, Pl_K its set of equivalence classes of absolute values and ℓ a prime number, we associate to K two $\mathbb{Z}_\ell$-modules, the first one of global nature and the second one local:
We define the ℓ-group of principal idèles as

$$\mathscr{R}_K = \mathbb{Z}_\ell \otimes_{\mathbb{Z}} K^\times, \tag{1}$$

and the ℓ-group of generalized idèles as

$$\mathscr{J}_K = \prod_{\mathfrak{p}\in \mathrm{Pl}_K}^{*} \mathscr{K}_\mathfrak{p}^\times, \tag{2}$$

where $\mathscr{K}_\mathfrak{p}$ is the ℓ-profinite completion of $K_\mathfrak{p}^\times$, i.e.

$$\mathscr{K}_\mathfrak{p}^\times = \varprojlim_k K_\mathfrak{p}^\times / K_\mathfrak{p}^{\times \ell^k}. \tag{3}$$

The product on (2) is restricted in the sense that for $(x_\mathfrak{p})_{\mathfrak{p}\in \mathrm{Pl}_K} \in \mathscr{J}_K$, almost all $x_\mathfrak{p}$ are units but a finite number of them. I will outline the algebraic and topological structure of these $\mathbb{Z}_\ell$-modules.

The natural map of $\mathscr{R}_K$ into $\mathscr{J}_K$, induced by the diagonal mapping of $K^\times$ into its group of idèles $\mathbb{I}_K$, is a continuous $\mathbb{Z}_\ell$-module monomorphisme. We define the ℓ-class group of idèles as

$$\mathscr{C}_K = \mathscr{J}_K / \mathscr{R}_K. \tag{4}$$

We can establish a correspondence between classical algebraic objects and the objects introduced above. In particular the ℓ-class group of divisors of K is isomorphic to the quotient of the ℓ-class group of idèles by the unit class subgroup:

$$Cl_K \simeq \mathscr{J}_K / \mathscr{R}_K \mathscr{U}_K, \tag{5}$$

from this fact it turns out that $\mathscr{C}_K$ is a compact $\mathbb{Z}_\ell$-module. Where

$$\mathscr{U}_K = \prod_{\mathfrak{p}\in \mathrm{Pl}_K} \mathscr{U}_\mathfrak{p}. \tag{6}$$

Furthermore, the ℓ-adic class field theory establishes correspondences between the Galois group of certain ℓ-extensions of K and subgroups of $\mathscr{C}_K$. For instance, the quotient $\mathscr{C}_K = \mathscr{J}_K / \mathscr{R}_K$ is isomorphic as a topological $\mathbb{Z}_\ell$-module to the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ of the maximal abelian ℓ-extension K^{ab} de K.

One might wonder whether the product formula for the absolute values still holds for the ℓ-adification of the idèle group, in fact this is the case if we replace the usual absolute values by the following construction:

For an element $x_\mathfrak{p}$ in $\mathscr{K}_\mathfrak{p}^\times$, we denote the ℓ-adic principal absolute value $|x_\mathfrak{p}|_\mathfrak{p}$, as the element of the multiplicative $\mathbb{Z}_\ell$-module U_ℓ^1 of principal units defined as follows:

$$|x_\mathfrak{p}|_\mathfrak{p} = \begin{cases} 1, \text{ if } \mathfrak{p} \text{ is complexe;} \\ \langle \mathrm{sg}(x_\mathfrak{p}) \rangle, \text{ if } \mathfrak{p} \text{ is real;} \\ \langle N\mathfrak{p}^{-v_\mathfrak{p}(x)} \rangle, \text{ if } \mathfrak{p} \text{ is ultrametric and prime to } \ell; \\ \langle N_{K_\mathfrak{p}/\mathbb{Q}_p}(x) \cdot N\mathfrak{p}^{-v_\mathfrak{p}(x)} \rangle, \text{ if } \mathfrak{p} \text{ lies over } \ell. \end{cases}$$

Hence, we are in position to state that the mapping

$$\xi = (x_\mathfrak{p})_\mathfrak{p} \to \prod_{\mathfrak{p} \in \mathrm{Pl}_K} |x_\mathfrak{p}|_\mathfrak{p} = \|\xi\| \tag{7}$$

is a continuous $\mathbb{Z}_\ell$-morphism from the ℓ-group $\mathscr{J}_K$ of generalized idèles of the number fields K on the group U_ℓ^1 of principal units of $\mathbb{Z}_\ell$. Its kernel

$$\widetilde{\mathscr{J}_K} = \{(x_\mathfrak{p})_\mathfrak{p} \in \mathscr{J}_K \mid \prod_{\mathfrak{p} \in \mathrm{Pl}_K} |x_\mathfrak{p}|_\mathfrak{p} = 1\} \tag{8}$$

is a closed submodule of $\mathscr{J}_K$, which contains the ℓ-adic tensor $\mathscr{R}_K$ of the multiplicative group $K^\times$.

We denote

$$\widetilde{\mathscr{U}_K} = \prod_{\mathfrak{p} \in \mathrm{Pl}_K} \widetilde{\mathscr{U}_\mathfrak{p}} \tag{9}$$

the group of cyclotomic norms in $\mathscr{J}_K$, where $\widetilde{\mathscr{U}_\mathfrak{p}} = \ker |\cdot|_\mathfrak{p}$ is the group of cyclotomic norms in $\mathscr{K}_\mathfrak{p}$.

The ℓ-adic class field theory yields the following isomorphisms

$$\widetilde{\mathscr{U}_K}\mathscr{R}_K/\mathscr{R}_K \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{lc}}) \text{ and } \widetilde{\mathscr{J}_K}/\mathscr{R}_K \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{c}}), \tag{10}$$

where K^{lc} is the locally cyclotomic maximal ℓ-extension and K^{c} is the $\mathbb{Z}_\ell$-cyclotomic extension.

The quotient

$$\widetilde{\mathscr{C}\ell}_K = \widetilde{\mathscr{J}_K}/\widetilde{\mathscr{U}_K}\mathscr{R}_K \tag{11}$$

is the ℓ-group of logarithmic classes of the number field K. It corresponds to the Galois group $\mathrm{Gal}(K^{\mathrm{lc}}/K^{\mathrm{c}})$.

The final goal of the talk is to explain how the ℓ-group of logarithmic classes is related to the Galois group $\mathscr{C}' = \mathrm{Gal}(C'_\infty/K_\infty)$ where $\mathscr{C}'_\infty$ is the Hilbert ℓ-class ℓ-group of the cyclotomic $\mathbb{Z}_\ell$-extension K_∞. Moreover, is intended to present the advances on the study of the non cyclotomic case.

References

[1] J-F. Jaulent, *L'arithmétique des ℓ-extensions*. (Thèse) - Pub. Math. Fac. Sci. Besançon Th. Nombres 1984-85 & 1985-86, fasc. 1, pp. 1-349 (1986).

[2] J-F. Jaulent, *Classes logarithmiques des corps de nombres*. Journal de Théorie Nombres Bordeaux 6, pp. 301-325, (1994).

[3] S. Lang, *Algebraic Number Theory*, Graduate Texts in Mathematics *110*, pp. 1-357 (1994).