An asymptotic distribution for $|L'/L(1,\chi)|$

Sumaia Saad Eddin*

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Abstract

Let χ be a Dirichlet character modulo q, let $L(s,\chi)$ be the attached Dirichlet L-function, and let $L'(s,\chi)$ denotes its derivative with respect to the complex variable s. The main purpose of this talk is to give an asymptotic formula for the $2k$-th power mean value of $|L'/L(1,\chi)|$ when χ ranges a primitive Dirichlet character modulo q for q prime. We derive some consequences, in particular a bound for the number of χ such that $|L'/L(1,\chi)|$ is large.

Keywords: Distribution function, Dirichlet L-function, Dirichlet characters.

1 Introduction and results

Let χ be a Dirichlet character modulo q, let $L(s,\chi)$ be the attached Dirichlet L-function, and let $L'(s,\chi)$ denotes its derivative with respect to the complex variable s. The values at 1 of Dirichlet L-series has received considerable attention, due to their algebraical or geometrical interpretation. Let us mention, in particular, the Birch and Swinnerton-Dyer conjectures, the Kolyvagin Theorem and the Gross-Stark conjecture. The BSD conjectures relate arithmetical problems about elliptic curves to analytic problems about the associated L-function. No proof of it is shown as of today (that is the million-dollar question!). These conjectures have been studied by many people and proved only in special cases, although there is extensive numerical evidence for their truth. The earliest results on the BSD conjectures are the Coates and Wiles result [7] for elliptic curves with complex multiplication, and the Gross-Zagier Theorem [14]. Related results are given in [15], [19] and [3]. The very recent paper in [1] shows that a positive proportion of elliptic curves have analytic rank 0; i.e., a positive proportion of elliptic curves have a non-vanishing L-function at $s=1$. Applying Kolyvagin's Theorem, it follows that a positive proportion of all elliptic curves satisfy the conjectures BSD. See also [4], [2], [6] and [8] for numerical evidence of these conjectures. A beautiful asymptotic formula of the fourth power moment in the q-aspect has been obtained by Health-Brown [11], for q prime. Recently, Bui and Heath-Brown [5] gave an asymptotic formula for the fourth power mean of Dirichlet L-functions averaged over primitive characters to modulus q and over $t \in [0,T]$ which is particularly effective when $q \geq T$ (see also [21] and [28]).

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Concerning the value of the derivative, Stark presented a conjecture for abelian L-functions with simple zeros at $s=0$, expressing the value of the derivative at $s=0$ in terms of logarithms of global units (see a series of papers [22], [23], [24] and [25]). Extending the conjectures of Stark, Gross stated in 1988 a relation between the derivative of the p-adic L-function associated to χ at its exceptional zero and the p-adic logarithm of a p-unit in the extension of a totally real field E cut out by χ, where χ is an abelian totally odd character of E. In 1996 Rubin [20] gave an extension of the conjectures of Stark, attempting to understand the values $L^{(r)}(\chi,0)$ when the order of vanishing r may be greater than one. In [10], the authors proved the conjecture of Gross when E is a real quadratic field and χ is a narrow ring class character. Recently, Ventullo [26] proposed an unconditional proof of the Gross-Stark conjecture in rank one.

Less is known about L'/L evaluated also at the point $s=1$, through these values are known to be fundamental in studying the distribution of primes since Dirichlet in 1837.

In this talk, we show that the values $|L'/L(1,\chi)|$ behave according to a distribution law. Let us state this result formally.

Theorem 1. *There exists a unique probability measure μ such that every continuous function f, we have*

$$\frac{1}{q-2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} f\left(\left|\frac{L'}{L}(1,\chi)\right|\right) \underset{q\to+\infty}{\longrightarrow} \int_0^{+\infty} f(t)\,d\mu(t), \tag{1}$$

where $\sum_{\chi \bmod q}$ denotes the summation over all the primitive characters $\chi \mod q$. Here the variable q ranges the odd primes.

We deduce the existence of μ by the general solution to the Stieltjes moment problem and the unicity by the criterion of Carleman.

This is an existence (and unicity) result, but getting an actual description of μ is a tantalizing problem. As we noted before, it is likely to have a geometrical or arithmetical interpretation, on which our approach gives no information.

The key to this result is to give an asymptotic formula of the $2k$-th power mean

$$\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \left|\frac{L'}{L}(1,\chi)\right|^{2k} \tag{2}$$

Under the generalized Riemann hypothesis (and later in [13] unconditionally), Ihara and Matsumoto [12] gave a stronger result related to the value-distributions of $\{L'/L(s,\chi)\}_\chi$ and of $\{\zeta'/\zeta(s+i\tau)\}_\tau$, where χ runs over Dirichlet characters with prime conductors and τ runs over $\mathbb{R}$.

A similar study on $L(1,\chi)$ has been partially achieved; when $k=1$ by Walum [27]. The former based on the Fourier series to evaluate $\sum|L(1,\chi)|^2$ for χ ranges the odd characters modulo a prime number. In 1989, Zhang [29] obtained an exact formula for general integer $q\geq 3$. More recently, Louboutin [16] gave an exact formula for the twisted moments. His result generalizes previous works. For general k, Zhang and Weiqiong [30] gave an exact calculating formula for the $2k$-th power mean of L-functions with $k \geq 3$.

Here is our result.

Theorem 2. *Let $\epsilon > 0$ and let χ be a primitive Dirichlet character modulo prime q. For k is an arbitrary non-negative integer, we have*

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \left| \frac{L'}{L}(1,\chi) \right|^{2k} = (q-2) \sum_{m \geq 1} \frac{\left(\sum_{m = m_1 \cdot m_2 \cdots m_k} \Lambda(m_1) \cdots \Lambda(m_k) \right)^2}{m^2} + \mathcal{O}_\epsilon \left(q^{9/10+\epsilon} \right),$$

where Λ is the Von Mangoldt's function.

The error term is not effective as we use the full force of Siegel's theorem on exceptional zeros.

We use an analytical method that is already used in [18] in contrast with the previous works that used essentially only elementary and combinatorial arguments. We introduce many simplifications in their intricate combinatorial argument, which is why we can handle the case of general k.

This result is strong in two aspects: it is valid for general k and we save a power of q (we did not try to optimize this saving). The method of proof relies in particular on a suitable average density estimate for the zeros of Dirichlet L-functions. Let us note that this approach does not work for $\frac{L'}{L}(1+it,\chi)$ when $t \neq 0$.

We also computed

$$\sum_{m \geq 1} \frac{\left(\sum_{m = m_1 \cdot m_2} \Lambda(m_1)\Lambda(m_2) \right)^2}{m^2} = 0.80508\cdots,$$

and

$$\sum_{m \geq 1} \frac{\left(\sum_{m = m_1 \cdots m_4} \Lambda(m_1) \cdots \Lambda(m_4) \right)^2}{m^2} = 1.242\cdots.$$

Here are some consequences of our main Theorem.

Corollary 1. *There exists $a > 0$ and $c > 0$ such that*

$$\liminf_q \frac{1}{q-2} \# \left\{ \chi \neq \chi_0 \ mod \ q \ ; \ \left| \frac{L'}{L}(1,\chi) \right| \leq a \right\} \leq 1 - c.$$

Since we can control all the moments, we have an estimation for the size of the tail of our distribution.

Corollary 2. *For $t \geq 1$. We have*

$$\liminf_q \frac{1}{q-2} \# \left\{ \chi \neq \chi_0 \ mod \ q \ ; \ \left| \frac{L'}{L}(1,\chi) \right| \geq t \right\} \ll e^{-\sqrt{t}/2}.$$

We do not know whether this bound can be improved upon.

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Sumaia Saad Eddin
Institute of Financial Mathematics
and Applied Number Theory,
Johannes Kepler University Linz,
Altenbergerstrasse 69, 4040 Linz, Austria.
sumaia.saad_eddin@jku.at