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The ℓ-adic Hasse norm principle

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Let ℓ be a fixed prime number, the goal of this talk is to present the ℓ-adic Hasse norm principle, to give the expression of its defect group and its arithmetic interpretation. We will then focus applications.

The objects I work with are those of ℓ-adic class field theory built by Jaulent [4]:

for a local field $K_\mathfrak{p}$ with maximal ideal $\mathfrak{p}$ and uniformizer $\pi_\mathfrak{p}$, we let:

$\mathcal{R}_{K_\mathfrak{p}} = \varprojlim_k K_\mathfrak{p}^\times / K_\mathfrak{p}^{\times \ell^k}$: the ℓ-adification of the multiplicative group of a local field, endowed with the logarithmic valuation $\tilde{v}_\mathfrak{p}$ [4]

$\mathcal{U}_{K_\mathfrak{p}} = \varprojlim_k U_\mathfrak{p} / U_\mathfrak{p}^{\ell^k}$: the ℓ-adification of the group of units $U_\mathfrak{p}$ of $K_\mathfrak{p}$

for a number field K we let:

$\mathcal{R}_K = \mathbb{Z}_\ell \otimes_\mathbb{Z} K^\times$: the ℓ-adic group of principal ideles

$\mathcal{J}_K = \prod^{res}_{\mathfrak{p} \in Pl_K} \mathcal{R}_{K_\mathfrak{p}}$: the ℓ-adic idele group

$\mathcal{U}_K = \prod_{\mathfrak{p} \in Pl_K} \mathcal{U}_{K_\mathfrak{p}}$: the subgroup of units

$\mathcal{C}_K = \mathcal{J}_K / \mathcal{R}_K$: the ℓ-adic idele class group

The starting point of this talk is the ℓ-adic Hasse norm principle.

Theorem 0.0.1. *The ℓ-adic Hasse norm principle [5]*

Let L/K be a cyclic ℓ-extension then a principal idele $x \in \mathcal{R}_L$ is a norm globally if and only if it is a norm everywhere locally i.e. for every completion $L_\mathfrak{P}/K_\mathfrak{p}$.

Thus we naturally introduce the group of defect of this principle defined as the quotient of the elements which are everywhere locally a norm denoted $\mathcal{N}_{L/K}$ over the group of global norms $N_{L/K}\mathcal{R}_L$. We want to give an arithmetic interpretation of this group.

Before going further, let's introduce the notion of logarithmic ramification and logarithmic divisors.

Definition 1. *Absolute and relative indexes : [4, def.1.3]*

Let K, L be number fields, $\mathfrak{p}$ a prime of K above p and $\mathfrak{P}$ a prime of L lying above $\mathfrak{p}$. Let's denote $\widehat{\mathbb{Q}_p^c}$ the $\widehat{\mathbb{Z}}$-cyclotomic extension of $\mathbb{Q}_p$, i.e. the compositum of all $\mathbb{Z}_q$-cyclotomic extensions of $\mathbb{Q}_p$ for all primes q and $\widehat{K_\mathfrak{p}^c}$ the compositum of $K_\mathfrak{p}$ and $\widehat{\mathbb{Q}_p^c}$.

i) the absolute and relative logarithmic ramification index of $\mathfrak{p}$ are respectively:

$$\tilde{e}_\mathfrak{p} = [K_\mathfrak{p} : \hat{\mathbb{Q}}_\mathfrak{p}^c \cap K_\mathfrak{p}] \qquad \tilde{e}_{L_\mathfrak{P}/K_\mathfrak{p}} = [L_\mathfrak{P} : \hat{K}_\mathfrak{p}^c \cap L_\mathfrak{P}]$$

ii) the absolute and relative logarithmic inertia degree of $\mathfrak{p}$ are respectively:

$$\tilde{f}_\mathfrak{p} = [\hat{\mathbb{Q}}_\mathfrak{p}^c \cap K_\mathfrak{p} : \mathbb{Q}_\mathfrak{p}] \qquad \tilde{f}_{L_\mathfrak{P}/K_\mathfrak{p}} = [\widehat{K_\mathfrak{p}^c} \cap L_\mathfrak{P} : K_\mathfrak{p}]$$

iii) $K/\mathbb{Q}$ is said logarithmically unramified at $\mathfrak{p}$ if $\tilde{e}_{\mathfrak{p}} = 1$, which means $K_{\mathfrak{p}} \subseteq \widehat{\mathbb{Q}_p^c}$.

iv) L/K is said logarithmically unramified at $\mathfrak{p}$ if $\tilde{e}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} = 1$, which implies $L_{\mathfrak{P}} \subseteq \widehat{K_{\mathfrak{p}}^c}$.

v) the degree of a prime $\mathfrak{p}$ is $\tilde{f}_{\mathfrak{p}} p$

Definition 2. *Let's define the following map*

$$\begin{aligned} div : \mathscr{J}_K &\longrightarrow \mathscr{D}\ell_K \\ \alpha = (\alpha_{\mathfrak{p}}) &\longmapsto div(\alpha) = \prod_{\textit{place finie de } K} \mathfrak{p}^{\widetilde{v_{\mathfrak{p}}}(\alpha_{\mathfrak{p}})} \end{aligned}$$

The group of logaritmic divisors of K is: $\mathscr{D}\ell_K = div(\mathscr{J}_K)$

The subgroup of logarithmic divisors of trivial degree is $\tilde{\mathscr{D}}\ell_{L/K}$

The subgroup of principal divisors is: $\mathscr{P}\ell_K = div(\mathscr{R}_K)$

The logarithmic class group of trivial degree is:

$$\tilde{\mathscr{C}}\ell_{L/K} = \tilde{\mathscr{D}}\ell_{L/K} / \mathscr{P}\ell_{L/K}$$

Definition 3. *Let's consider those definitions*

- *the logarithmic inertia subgroup associated to a prime $\mathfrak{p}$, denoted $\widetilde{\Gamma}_{L/K,\mathfrak{p}}$, is the subgroup of the decomposition subgroup of $\mathfrak{p}$ which fixes the maximal logarithmically unramified extension of K*
- *Let's consider the following set of the inertia subgroup $\widehat{\Gamma}_{L/K} = \{\sigma \in \prod_{\mathfrak{p}|\tilde{\mathfrak{f}}_{L/K}} \widetilde{\Gamma}_{L/K,\mathfrak{p}}$ such that $\prod_{\mathfrak{p}|\tilde{\mathfrak{f}}_{L/K}} \sigma_{\mathfrak{p}} = 1\}$.*

The fundamental theorem is

Theorem 0.0.2. *The index of the group of defect [8]Th.3.1.1*

Let L/K be a finite and abelian ℓ-extension,

let $\tilde{\mathscr{C}}\ell_L^*$ be the kernel of the norm map $N_{L/K} : \tilde{\mathscr{C}}\ell_L \longrightarrow N_{L/K}\tilde{\mathscr{C}}\ell_L$

$\Delta_{L/K}$ the ideal augmentation of the Galois group of L/K

and $\tilde{\mathscr{E}}_K = \{x \in \mathscr{R}_K / \tilde{v}_{\mathfrak{p}}(x) = 0\}$ the group of logarithmic units, then we get:

$$|\hat{\Gamma}_{L/K}|(\mathscr{N}_{L/K} : N_{L/K}\mathscr{R}_L) = (\tilde{\mathscr{C}}\ell_L^* : \tilde{\mathscr{C}}\ell_L^{\Delta_{L/K}})(\tilde{\mathscr{E}}_K : \tilde{\mathscr{E}}_K \cap N_{L/K}\mathscr{R}_L)$$

The main tool to prove this theorem is the logarithmic Hasse symbol [8].

Let's now focus on applications of this theorem.

By [4, Section4], we have

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & \tilde{P}\ell_K & \longrightarrow & \tilde{D}\ell_K & \longrightarrow & \tilde{\mathscr{C}}\ell_K & \longrightarrow & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{j}} & & & & \\ 1 & \longrightarrow & \tilde{P}\ell_L^G & \longrightarrow & \tilde{D}\ell_L^G & \longrightarrow & \tilde{\mathscr{C}}\ell_L^G & \longrightarrow & \mathrm{H}^1(G, \tilde{P}\ell_L) & \longrightarrow & \mathrm{H}^1(G, \tilde{D}\ell_L) \end{array}$$

where $\tilde{j}$ denotes the extension morphism.

Using the snake's lemma, we thus get

$$1 \to \tilde{P}\ell_L^G / \tilde{P}\ell_K \to \tilde{D}\ell_L^G / \tilde{D}\ell_K \to \tilde{\mathscr{C}}\ell_L^G / \tilde{j}(\tilde{\mathscr{C}}\ell_K) \to \mathrm{H}^1(G, \tilde{P}\ell_L) \xrightarrow{\phi} \mathrm{H}^1(G, \tilde{D}\ell_L).$$

Proposition 0.0.1. *Application [8]Prop.3.3.2*

Let L/K be a cyclic ℓ-extension of Galois group G, satisfaying the Gross's conjecture, we get:

$$(\tilde{\mathscr{C}}\ell_K : N_{L/K}\tilde{\mathscr{C}}\ell_L) = \frac{|\hat{\Gamma}_{L/K}|[L^c : K^c]}{\prod_{\mathfrak{p}\in P\ell_K^\infty} d_\mathfrak{p}(L/K)\ \prod_{\mathfrak{p}\in P\ell_K^0} \tilde{e}_\mathfrak{p}(L/K)\ |\mathrm{Coker}\phi|}$$

Another application of this arithmetic interpretation are the interesting relations we get if we assume that L/K is a cyclic extension such that $|\hat{\Gamma}_{L/K}| = 1$:

Proposition 0.0.2. *Application [8]Section 3.4*

Let L/K be a cyclic ℓ-extension such that $|\hat{\Gamma}_{L/K}| = 1$, then the previous theorem gives

$$(\tilde{\mathscr{C}}\ell_L^* : \tilde{\mathscr{C}}\ell_L^{\Delta_{L/K}})(\tilde{\mathscr{E}}_K : \tilde{\mathscr{E}}_K \cap N_{L/K}\mathscr{R}_L) = 1.$$

Thus, we obtain the following relations:

$$\tilde{\mathscr{C}}\ell_L^* = \tilde{\mathscr{C}}\ell_L^{\Delta_{L/K}} \qquad \tilde{\mathscr{C}}\ell_L^G = N_{L/K}\tilde{\mathscr{C}}\ell_L \qquad \tilde{\mathscr{E}}_K \subseteq N_{L/K}\mathscr{R}_L.$$

We will then, focus on an explicit example of this theorem.

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