On the dispersion of the zeros of the partial sums of the Riemann zeta function

Gaspar Mora

Departamento de Análisis Matemático. Facultad de Ciencias. Universidad de Alicante. Spain

ABSTRACT

To date, the first ten trillions of non-trivial zeros of the Riemann zeta function lie on the line $x = 1/2$ such as Riemann conjectured in 1859. However, it will be shown in this talk that the zeros of its partial sums

$$\zeta_n(z) = \sum_{k=1}^{n} 1/k^z, \quad n \geq 2, \quad z = x + iy,$$

except for $\zeta_2(z)$ whose zeros are aligned on the imaginary axis, are dispersed of a uniform manner, with respect to the real parts, on vertical strips of the complex plane having a width which tends to infinity as n does. Moreover, it will be also showed a formula to determine the lower bound of every strip. To do it we will consider the two following facts:

a) For each $n \geq 3$, the strip that contains to the infinitely many zeros of $\zeta_n(z)$ is bounded by the lines of equations $x = a_{\zeta_n(z)}$, $x = b_{\zeta_n(z)}$, where

$$a_{\zeta_n(z)} := \inf \{\Re z : \zeta_n(z) = 0\}, \quad b_{\zeta_n(z)} := \sup \{\Re z : \zeta_n(z) = 0\}. \tag{1}$$

For the bound $a_{\zeta_n(z)}$ which has been defined in (1), we have the estimate

$$a_{\zeta_{n+2}(z)} = -\frac{\log 2}{\log\left(1 + \frac{1}{n}\right)} + \Delta_{n+2}, \quad n \geq 1, \tag{2}$$

with Δ_{n+2} such that

$$\limsup_{n\to\infty} |\Delta_{n+2}| \leq \log 2, \tag{3}$$

found by G. Mora [12] in 2015 after to study the papers of Borwein, Fee, Ferguson and van der Waall [2], on one hand, and Balazard and Velásquez-Castañón [1], on the other hand. Indeed, in 2007 Borwein et al. [2] gave the first estimate (stated without proof) of $a_{\zeta_n(z)}$ by means of the formula

$$\zeta_n(z) = -(n - 3/2)\log 2, \tag{4}$$

after a hard computation. A little later, in 2009, Balazard and Velásquez-Castañón [1] proved that

$$\lim_{n\to\infty} \frac{a_{\zeta_n(z)}}{n} = -\log 2. \tag{5}$$

Then, since $\lim_{n\to\infty}(n + a)\log\left(1 + \frac{1}{n}\right) = 1$ for any $a \in \mathbb{R}$, our result (2), firstly, confirms that the estimate computationally obtained by Borwein et al. is asymptotically valid. Second, our estimate (2) implies in particular the above relation (5) due to Balazard and Velásquez Castañón.

The upper bound $b_{\zeta_n(z)}$, defined in (1), was estimated by means of the formula

$$b_{\zeta_n(z)} = 1 + \left(\frac{4}{\pi} - 1 + o(1)\right)\frac{\log\log n}{\log n}, \quad n \geq 3, \tag{6}$$

by H.L. Montgomery and R.C. Vaughan [5] in 2001 by completing a previous work of Montgomery [4] in 1983. This last paper was decisive to demonstrate that an important result of Turán [13] of 1948, where he had settled a connexion between a particular distribution of the zeros of the partial sums near the line $x = 1$ and the Riemann Hypothesis, was vacuous.

From (2)-(3) and (6), it is immediate that

$$\lim_{n\to\infty} a_{\zeta_n(z)} = -\infty, \quad \lim_{n\to\infty} b_{\zeta_n(z)} = 1. \tag{7}$$

Therefore, from (7) it follows that the widths, $b_{\zeta_n(z)} - a_{\zeta_n(z)}$, of the critical strips, that contain the zeros of the partial sums of the Riemann zeta function, form a sequence satisfying

$$\lim_{n\to\infty} \left(b_{\zeta_n(z)} - a_{\zeta_n(z)}\right) = \infty.$$

b) As a consequence of the distribution of the prime numbers in the natural series, we will deduce that the zeros of every $\zeta_n(z)$ are situated in its corresponding critical strip of a particular way with respect to the real part, at least asymptotically. That is, in [8] we proved, by using the prime number theorem, the existence of a positive integer N such that for any n, greater or equal than N, the set of real parts of the zeros of $\zeta_n(z)$ is dense in each interval $\left[a_{\zeta_n(z)}, b_{\zeta_n(z)}\right]$. It means that given an arbitrary line contained in the critical strip of $\zeta_n(z)$, this function possesses infinitely many zeros arbitrarily close to that line, for every $n \geq N$. Equivalently, for each $n \geq N$, any vertical strip of arbitrary width contained in $\left[a_{\zeta_n(z)}, b_{\zeta_n(z)}\right] \times \mathbb{R}$ is a region which is not zero-free of $\zeta_n(z)$.

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