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A space of weight 1 modular forms attached to totally real cubic number fields.

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The main goal of this talk is to exhibit a canonical subspace of a space of weight 1 modular forms that is parametrized by the set of isomorphism classes of cubic fields of a fixed fundamental discriminant. In case that the fields have ramification at infinity, the construction is well known. Here, using the theory of integral traces, I will show how to construct such a subspace for cubic fields with no ramification at infinity i.e., totally real cubic fields.

Let δ be a fundamental discriminant, and let $\mathscr{C}_\delta$ be the set of isomorphism classes of cubic number fields of discriminant δ. Recall that an integer is called *fundamental discriminant* if it is equal to the discriminant of a quadratic number field. Let N be a positive integer and let ε be a Dirichlet character modulo N. Let $\mathscr{M}_1\left(\Gamma_0(N),\varepsilon\right)$ be the space of weight 1 modular forms of level N and nebentypus ε. Suppose $\mathscr{C}_\delta \neq \emptyset$ and that $\delta < 0$, i.e. cubic fields with at least one complex place. Then, associated to each $K \in \mathscr{C}_\delta$ there is a weight 1 modular form

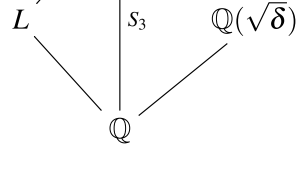
$$\mathfrak{f}_K \in \mathscr{M}_1\left(\Gamma_0(|\delta|), \left(\frac{\delta}{\cdot}\right)\right)$$

such that:

1. the map $K \mapsto \mathfrak{f}_K$ is injective.
2. the set $\{\mathfrak{f}_K : K \in \mathscr{C}_\delta\}$ is a linearly independent subset of $\mathscr{M}_1\left(\Gamma_0(|\delta|), \left(\frac{\delta}{\cdot}\right)\right)$.

The above follows from a particular case of Weil-Langlands converse theorem. If instead of considering cubic fields ramified at infinity we consider totally real cubic fields, then it is not possible to apply Weil-Langlands to produce modular forms. The point is that in the totally real case the Galois representations involved are even.

Let L be a cubic number field with discriminant $\delta < 0$, which is fundamental, and let $\widetilde{L}$ be its Galois closure.



By identifiying S_3 with $\mathrm{Gal}(\widetilde{L}/\mathbb{Q})$, and considering the irreducible 2-dimensional representation of S_3, one obtains an irreducible dihedral representation of $\rho_L : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$. Such a representation is induced by a non-trivial representation of H. Since d is a fundamental discriminat $\widetilde{L}$ is contained in the Hilbert class field of $\mathbb{Q}(\sqrt{\delta})$. Thus, by the conductor formula for induced representations, we have that ρ_L has conductor $|\delta|$. From the above diagram we see that $\det(\rho_L)$ is the inflation of the non-trivial character of $\mathrm{Gal}(\mathbb{Q}(\sqrt{\delta})/\mathbb{Q})$. In other words, for all primes p not dividing d we have that $\det(\rho_L(\mathrm{Frob}_p)) = \left(\frac{\delta}{p}\right)$. In particular, ρ_L is an odd representation. Let $L(s,\rho_L) = \sum_{n=1} \frac{a_n}{n^s}$ be the Artin L-series attached to ρ_L and let $\mathfrak{f}_L = \sum_{n=1} a_n q^n$ be the q-expansion of $L(s,\rho_L)$. It follows from the theory of Theta series that

$$\mathfrak{f}_L \in \mathscr{M}_1\left(\Gamma_0(|\delta|), \left(\frac{\delta}{\cdot}\right)\right).$$

Moreover, each $\mathfrak{f}_L$ is a normalized cuspidal eigenform.

Theorem 0.1. *The following is injective:*

$$\begin{array}{rccc} \Phi: & \mathscr{C}_\delta & \to & \mathscr{M}_1\left(\Gamma_0(|\delta|), \left(\frac{\delta}{\cdot}\right)\right) \\ & L & \mapsto & \mathfrak{f}_L \end{array}$$

In this talk we explain how to, for totally real cubic fields of fundamental discriminant, give an alternative construction of weight 1 modular forms satisfying properties (1) and (2) above. The main result to be explained in the talk is: Given K a cubic field with positive fundamental discriminant d there is a number d_3, depending only on d, and weight 1 modular form $\Theta(K) := f_K$ such that

Theorem 0.2. *Let d be a positive fundamental discriminant, and suppose that $\mathscr{C}_d \neq \emptyset$. Then,*

$$\begin{array}{rccc} \Theta: & \mathscr{C}_d & \to & \mathscr{M}_1\left(\Gamma_0(|d_3|), \left(\frac{d_3}{\cdot}\right)\right) \\ & K & \mapsto & f_K \end{array}$$

is injective.