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Higher Mahler measure of some n-variable families

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We prove formulas for the k-higher Mahler measure of a family of rational functions with an arbitrary number of variables. Our formulas reveal relations with multiple polylogarithms evaluated at certain roots of unity.

For k a positive integer, the k-higher Mahler measure of a non-zero, n-variable, rational function $P(x_1,\dots,x_n)\in\mathbb{C}(x_1,\dots,x_n)$ is given by

$$\mathrm{m}_k\left(P(x_1,\dots,x_n)\right)=\int_0^1\dots\int_0^1\log^k\left|P\left(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n}\right)\right|d\theta_1\dots d\theta_n.$$

We observe that the case $k=1$ recovers the formula for the "classical" Mahler measure. This function, originally defined as a height on polynomials, has attracted considerable interest in the last decades due to its connection to special values of the Riemann zeta function, and of L-functions associated to objects of arithmetic significance such as elliptic curves as well as special values of polylogarithms and other special functions. Part of such phenomena has been explained in terms of Beilinson's conjectures via relationships with regulators by Deninger [2] and others.

Deninger remarked that k-higher Mahler measures are expected to yield different regulators than the ones that appear in the case of the usual Mahler measure and they may reveal a more complicated structure at the level of the periods (see [4] for more details). This motivated us to follow a program of understanding of periods arising from k-higher Mahler measure. An essential part of this program is the generation of examples of formulas of k-higher Mahler measure involving special functions that can be easily expressed as periods, such as polylogarithms.

The k-higher Mahler measure is naturally associated to another function called the Zeta Mahler measure. For P as before, the Zeta Mahler measure is defined by

$$Z(s,P):=\int_0^1\dots\int_0^1\left|P\left(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n}\right)\right|^s d\theta_1\dots d\theta_n.$$

The relationship is given by

$$Z(s,P)=\sum_{k=0}^{\infty}\frac{\mathrm{m}_k(P)s^k}{k!}.$$

Let

$$R_m(x_1,\dots,x_m,z):=z+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right).$$

be a rational function in the $m+1$ variables $x_1,\dots,x_m,z$.

In [3], the Mahler measure of the above family is computed. The techniques of that work are combined with results on the Zeta Mahler measure obtained by Akatsuka [1] in order to prove formulas for $\mathrm{m}_k(R_m)$ involving the Riemann zeta function, the Dirichlet L-function in the character of conductor 4, and multiple polylogarithms evaluated at roots of unity.

More precisely, in [5] we obtain the following result.

Theorem 1. *We have, for $n \geq 1$, the following formula*

$$\mathrm{m}_k(R_{2n}) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2, 4^2, \ldots, (2n-2)^2)}{(2n-1)!} \left(\frac{2}{\pi}\right)^{2h} \mathscr{A}_k(h),$$

where

$$\mathscr{A}_k(h) := (2h+k-1)! \left(1 - \frac{1}{2^{2h+k}}\right) \zeta(2h+k) + (-1)^k k! \sum_{\frac{k}{2}-1 \leq n \leq k-2} \frac{1}{2^{k-n-1}} \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_n)\in\{1,2\}^n \\ \#\{i:\varepsilon_i=2\}=k-n-2}} (2h-1)! \mathscr{L}_{\varepsilon_1,\ldots,\varepsilon_n,2,2h}(1,\ldots,1)$$

$$+ \sum_{j=1}^{k-2} \frac{(-1)^{k+j} k!}{j!} \sum_{\frac{k-j}{2}-1 \leq n \leq k-j-2} \frac{1}{2^{k-n-j}} \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_n)\in\{1,2\}^n \\ \#\{i:\varepsilon_i=2\}=k-j-n-2}} (2h+j-1)! \mathscr{L}_{\varepsilon_1,\ldots,\varepsilon_n,2,2h+j}(1,\ldots,1).$$

We have, for $n \geq 0$, the following formula

$$\mathrm{m}_k(R_{2n+1}) = \sum_{h=0}^{n} \frac{s_{n-h}(1^2, 3^2, \ldots, (2n-1)^2)}{(2n)!} \left(\frac{2}{\pi}\right)^{2h+1} \mathscr{B}_k(h)$$

where

$$\mathscr{B}_k(h) := (2h+k)! L(\chi_{-4}, 2h+k+1) + (-1)^{k+1} k! \sum_{\frac{k}{2}-1 \leq n \leq k-2} \frac{1}{2^{k-n-1}} \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_n)\in\{1,2\}^n \\ \#\{i:\varepsilon_i=2\}=k-n-2}} i(2h)! \mathscr{L}_{\varepsilon_1,\ldots,\varepsilon_n,2,2h+1}(1,\ldots,1,i,i)$$

$$+ \sum_{j=1}^{k-2} \frac{(-1)^{k+j+1} k!}{j!} \sum_{\frac{k-j}{2}-1 \leq n \leq k-j-2} \frac{1}{2^{k-n-j}} \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_n)\in\{1,2\}^n \\ \#\{i:\varepsilon_i=2\}=k-j-n-2}} i(2h+j)! \mathscr{L}_{\varepsilon_1,\ldots,\varepsilon_n,2,2h+j+1}(1,\ldots,1,i,i).$$

Here,

$$\mathscr{L}_{n_1,\ldots,n_m}(w_1,\ldots,w_m) = \sum_{(r_1,\ldots,r_m)\in\{0,1\}^m} (-1)^{r_m} \mathrm{Li}_{n_1,\ldots,n_m}((-1)^{r_1} w_1, \ldots, (-1)^{r_m} w_m), \tag{1}$$

given by combinations of multiple polylogarithms (of length m), defined for n_i positive integers by

$$\mathrm{Li}_{n_1,\ldots,n_m}(w_1,\ldots,w_m) = \sum_{0<j_1<\cdots<j_m} \frac{w_1^{j_1} \cdots w_m^{j_m}}{j_1^{n_1} \cdots j_m^{n_m}}.$$

The series above is absolutely convergent for $|w_i| \leq 1$ and $n_m > 1$.

Finally, for $a_1, \ldots a_m \in \mathbb{C}$, consider

$$s_\ell(a_1,\ldots,a_m) = \begin{cases} 1 & \text{if } \ell = 0, \\ \sum_{i_1<\cdots<i_\ell} a_{i_1} \cdots a_{i_\ell} & \text{if } 0 < \ell \leq m, \\ 0 & \text{if } m < \ell. \end{cases}$$

For the sake of clarity, we record here the case of $k = 2$.

$$\mathscr{A}_2(h) := (2h+1)! \left(1 - \frac{1}{2^{2h+2}}\right) \zeta(2h+2) + (2h-1)! \mathscr{L}_{2,2h}(1,1)$$
$$\mathscr{B}_2(h) := (2h+2)! L(\chi_{-4}, 2h+3) - i(2h)! \mathscr{L}_{2,2h+1}(i,i)$$

The case of $k = 1$ clearly yields formulas that only depend on $\zeta(s)$, $L(\chi_{-4}, s)$ and powers of π. This is equivalent to saying that all the terms can be expressed in terms of polylogarithms of length one. There is another case in which we can prove a similar formula.

Corollary 2. *The previous result includes the following particular case*

$$\mathrm{m}_2(R_2) = -\frac{31\pi^2}{360} + \frac{28}{\pi^2}\log 2\zeta(3) + \frac{32}{\pi^2}\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{4}{3\pi^2}\log^2 2(\log^2 2 - \pi^2), \tag{2}$$

where all the terms are product of polylogarithms of length one.

In this talk, we will give more context over the significance of Theorem 1, show the ideas behind its proof, and discuss possible extensions.

References

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