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Torsion of rational elliptic curve over number fields

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Let E be an elliptic curve defined over a number field K. The Mordell-Weil Theorem states that the set of K-rational points, $E(K)$, is a finitely generated abelian group. So it can be written as $E(K) \simeq E(K)_{\text{tors}} \oplus \mathbb{Z}^r$, for some non-negative integer r (rank of E) and some finite torsion subgroup $E(K)_{\text{tors}}$. It is well known that there exist two positive integers n, m such that $E(K)_{\text{tors}}$ is isomorphic to $\mathcal{C}_n \times \mathcal{C}_m$, where $\mathcal{C}_n$ be the cyclic group of order n

Let d be a positive integer. The set $\Phi(d)$ of possible torsion structures of elliptic curves defined over number fields of degree d has been deeply studied by several authors. The case $d = 1$ was obtained by Mazur [6, 7]:

$$\Phi(1) = \{\mathcal{C}_n \mid n = 1, \ldots, 10, 12\} \cup \{\mathcal{C}_2 \times \mathcal{C}_{2m} \mid m = 1, \ldots, 4\}.$$

The case $d = 2$ was completed by Kamienny [4] and Kenku and Momose [5]. There are not any other cases where $\Phi(d)$ has been completely determined.

Najman [8] has extended this study to the set $\Phi_{\mathbb{Q}}(d)$ of possible torsion structures over a number field of degree d of an elliptic curve defined over $\mathbb{Q}$. He has obtained a complete description of $\Phi_{\mathbb{Q}}(2)$ and $\Phi_{\mathbb{Q}}(3)$.

The objectives of this talk is to show recent results in this direction. Fix a possible torsion structure over $\mathbb{Q}$, say $G \in \Phi(1)$. We will study the sets:

- $\Phi_{\mathbb{Q}}(d, G)$ of possible groups that can appear as the torsion subgroup over any number field of degree d, of an elliptic curve E defined over the rationals, such that $E(\mathbb{Q})_{\text{tors}} = G$.
- $\mathcal{H}_{\mathbb{Q}}(d, G) = \{S_1, \ldots, S_n\}$ where, for any $i = 1, \ldots, n$, $S_i = [H_1, \ldots, H_m]$ is a list, with $H_i \in \Phi_{\mathbb{Q}}(d, G) \setminus \{G\}$, and there exists an elliptic curve E_i defined over $\mathbb{Q}$ such that:
  - $E_i(\mathbb{Q})_{tors} = G$.
  - There are number fields $K_1, \ldots, K_m$ (non–isomorphic pairwise) of degree d with $E_i(K_j)_{tors} = H_j$, for all $j = 1, \ldots, m$.

We give a complete description of the sets $\Phi_{\mathbb{Q}}(d, G)$ and $\mathcal{H}_{\mathbb{Q}}(d, G)$ for any $G \in \Phi(1)$ and $d = 2$ or $d = 3$.

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