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On the Selmer group associated to a modular form and an algebraic Hecke character.

Y. Elias[1]

[1] *McGill University, Montreal, Canada, yara.elias@mail.mcgill.ca*

Kolyvagin [5, 2] constructs an Euler system based on Heegner points and uses it to bound the size of the Selmer group of certain (modular) elliptic curves E over imaginary quadratic fields K assuming the non-vanishing of a suitable Heegner point. In particular, this implies that

$$\operatorname{rank}(E(K)) = 1,$$

and the Tate-Shafarevich group $Ш(E/K)$ is finite. Bertolini and Darmon adapt Kolyvagin's descent to Mordell-Weil groups over ring class fields [3]. More precisely, they show that for a given character χ of $\mathrm{Gal}(K_c/K)$ where K_c is the ring class field of K of conductor c,

$$\operatorname{rank}(E(K_c)^{\chi}) = 1,$$

assuming that the projection of a suitable Heegner point is non-zero. Nekovar [1] adapts the method of Euler systems to modular forms of higher even weight to describe the image by the Abel-Jacobi map Φ of Heegner cycles on the associated Kuga-Sato varieties, hence showing that

$$\dim_{\mathbb{Q}_p}(\mathrm{Im}(\Phi)\otimes\mathbb{Q}_p) = 1,$$

assuming the non-vanishing of a suitable Heegner cycle. In [7], we combined these two approaches to study modular forms of higher even weight twisted by ring class characters of imaginary quadratic fields and showed that

$$\dim\mathbb{Q}_p(\mathrm{Im}(\Phi)\otimes\mathbb{Q}_p) = 1,$$

assuming the non-vanishing of a suitable Heegner cycle. In this talk, after recalling the previous results, we present our study of the Selmer group associated to a modular form of even weight $r+2$ and an algebraic Hecke character ψ of infinity type $(r,0)$ [8]. The case of a Hecke character of infinity type $(0,0)$ corresponds to the setting of Nekovar's work [1] and its generalization in [7]. Our setting involves generalized Heegner cycles introduced by Bertolini, Darmon and Prasanna in [4].

Our motivation stems from the Beilinson-Bloch conjecture that predicts that

$$\dim_{\mathbb{Q}}(\mathrm{Im}(\Phi)\otimes\mathbb{Q}) = ord_{s=r+1}L(f\otimes\theta_{\psi},s),$$

where θ_{ψ} is the theta series associated to ψ [10, 11].

Let f be a normalized newform of level $N \geq 5$ and even weight $r+2 \geq 2$. Denote by $K = \mathbb{Q}(\sqrt{-D})$ an imaginary quadratic field with odd discriminant satisfying the Heegner hypothesis, that is primes dividing N split in K. For simplicity, we assume that $|\mathscr{O}_K^{\times}| = 2$. Let

$$\psi : \mathbb{A}_K^{\times} \longrightarrow \mathbb{C}^{\times}$$

be an algebraic Hecke character of K of infinity type $(r,0)$. Then there is an elliptic curve A defined over the Hilbert class field K_1 of K with complex multiplication by $\mathscr{O}_K$ such that ψ is the Hecke character associated to A by [9, Theorem 9.1.3]. Furthermore, A is a $\mathbb{Q}$-curve by the assumption on the parity of D, that is A is K_1- isogenous to its conjugates in $\mathrm{Aut}(K_1)$ (see [9, Section 11]). Consider a prime p satisfying

$$(p, ND\phi(N)N_A r!) = 1,$$

where N_A is the conductor of A. We denote by V_f the f-isotypic part of the p-adic étale realization of the motive associated to f by Deligne and by V_A the p-adic étale realization of the motive associated to A. Let $\mathscr{O}_F$ be the ring of integers of

$$F = \mathbb{Q}(a_1, a_2, \cdots, b_1, b_2, \cdots),$$

where the a_i's are the coefficients of f and the b_i's are the coefficients of θ_ψ. Then V_f and V_A give rise (by extending scalars appropriately) to free $\mathscr{O}_F \otimes \mathbb{Z}_p$-modules of rank 2. We denote by

$$V = V_f \otimes_{\mathscr{O}_F \otimes \mathbb{Z}_p} V_A(r+1)$$

the p-adic étale realization of the twisted motive associated to f and ψ and let $V_{\wp_1}$ be its localization at a prime $\wp_1$ in F dividing p. Then $V_{\wp_1}$ is a four dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We also denote by $\mathscr{O}_{F,\wp_1}$ the localization of $\mathscr{O}_F$ at $\wp_1$. By the Heegner hypothesis, there is an ideal $\mathscr{N}$ of $\mathscr{O}_K$ satisfying

$$\mathscr{O}_K/\mathscr{N} \simeq \mathbb{Z}/N\mathbb{Z}.$$

We can therefore fix level N structure on A, that is a point of exact order N defined over the ray class field L_1 of K of conductor $\mathscr{N}$. Consider a pair (φ_1, A_1) where A_1 is an elliptic curve defined over K_1 with level N structure and

$$\varphi_1 : A \longrightarrow A_1$$

is an isogeny over $\overline{K}$. We associate to it a codimension $r+1$ cycle on V

$$\Upsilon_{\varphi_1} = Graph(\varphi_1)^r \subset (A \times A_1)^r \simeq (A_1)^r \times A^r \subset W_r \times A^r$$

and define a *generalized Heegner cycle* of conductor 1

$$\Delta_{\varphi_1} = e_r \Upsilon_{\varphi_1},$$

where e_r is an appropriate projector. Then Δ_{φ_1} is defined over L_1. The Selmer group

$$S \subseteq H^1(L_1, V_{\wp_1}/p)$$

consists of the cohomology classes whose localizations at a prime v of L_1 lie in

$$\begin{cases} H^1(L_{1,v}^{ur}/L_{1,v}, V_{\wp_1}/p) \text{ for } v \text{ not dividing } NN_A p \\ H^1_f(L_{1,v}, V_{\wp_1}/p) \text{ for } v \text{ dividing } p \end{cases}$$

where $L_{1,v}$ is the completion of L_1 at v, and

$$H^1_f(L_{1,v}, V_{\wp_1}/p)$$

is the *finite part* of $H^1(L_{1,v}, V_{\wp_1}/p)$ [6, part 1]. We denote by $Fr(v)$ the Frobenius element generating $\mathrm{Gal}(L_{1,v}^{ur}/L_{1,v})$. In this talk, we will sketch a proof of the following theorem.

Theorem: Let p be such that

$$\mathrm{Gal}\left(L_1(V_{\wp_1}/p)/L_1\right) \simeq \mathrm{Aut}(V_{\wp_1}/p), \text{ and } (p, ND\phi(N)N_A r!) = 1.$$

Suppose that $V_{\wp_1}/p$ is a simple $\mathrm{Aut}(V_{\wp_1}/p)$-module. Suppose further that the eigenvalues of $Fr(v)$ acting on $V_{\wp_1}$ are not equal to 1 modulo p for v dividing NN_A. Assume $\Phi(\Delta_{\varphi_1}) \neq 0$ where

$$\Phi(\Delta_{\varphi_1}) \in H^1(L_1, V_{\wp_1}/p)$$

is the image by the Abel-Jacobi map of the generalized Heegner cycle Δ_{φ_1}. Then the Selmer group S has dimension 1 over $\mathscr{O}_{F,\wp_1}/p$.

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