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On sets free of sumsets with summands of prescribed size

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A popular topic in combinatorial/additive number theory is the study of extremal sets of integers free of subsets with some given particular shape. Given $r+1$ integers $r, \ell_1, \dots, \ell_r$, with $r \geq 1$ and $2 \leq \ell_1 \leq \cdots \leq \ell_r$, we focus our research in the shape of sumsets of r summands, where the s^{th} summand has size ℓ_s $(1 \leq s \leq r)$. In outline, we tackle extremal problems about sets that do not contain sumsets with summands of prescribed size. Our analysis includes both finite sets and infinite sequences. We also connect these problems with extremal problems on graphs that are free of complete r-partite sub-graphs.

Definition 1 *Given an abelian group G we say that $A \subset G$ is a $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$-free set if A does not contain any sumset of the form $L_1 + \cdots + L_r = \{\lambda_1 + \cdots + \lambda_r : \lambda_i \in L_i,\ i = 1, \dots, r\}$, with $|L_i| = \ell_i,\ i = 1, \dots, r$. For $r = 2$ we simply write $\mathscr{L}_{\ell_1,\ell_2}$.*

We remind two well known particular cases in this definition. Firstly note that the shape of the sumset $L_1 + L_2$, with $|L_1| = |L_2| = 2$, can be depicted as two traslates of L_1:

A Sidon set can be characterized as being free of this shape, that is to say the $\mathscr{L}_{2,2}$-free sets are the Sidon sets.

Secondly, a Hilbert cube of dimension r is a sumset of the form $L_1 + \cdots + L_r$ with $|L_1| = \cdots = |L_r| = 2$. Thus $\mathscr{L}^{(r)}_{2,\dots,2}$-free sets are those free of Hilbert cubes of dimension r. In the case of Hilbert cube of dimension 3, L_1 is translated twice and then the resulting structure is in turn translated twice, giving the following shape:

1 $\mathscr{L}$-free sets problems in intervals and finite abelian groups

Estimating the largest size of a set $A \subset \{1, \dots, n\}$ that is $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$-free is an interesting and significant problem.

Definition 2 *We will denote by $F\left(n, \mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}\right)$ the size of a largest $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$-free set in the interval $\{1, \dots, n\}$.*

Our first result is a general upper bound that recovers known upper bounds for several particular cases.

Theorem 1 *For any $r \geq 2$ and $2 \leq \ell_1 \leq \cdots \leq \ell_r$ we have*

$$F\left(n, \mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}\right) \leq (\ell_r - 1)^{\frac{1}{\ell_1 \cdots \ell_{r-1}}} n^{1 - \frac{1}{\ell_1 \cdots \ell_{r-1}}} + O\left(n^{\frac{1}{2} + \frac{1}{2\ell_{r-1}} - \frac{1}{\ell_1 \cdots \ell_{r-1}}}\right).$$

The probabilistic method provides a general lower bound for $F(n, \mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r})$.

Theorem 2 *For any $r \geq 2$ and $2 \leq \ell_1 \leq \cdots \leq \ell_r$ we have*

$$F(n, \mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}) \geq n^{1-\frac{\ell_1+\cdots+\ell_r\ -r}{\ell_1\cdots\ell_r\ -1}-o(1)}.$$

The exponents in Theorems 1 and 2 are distinct and to close the gap between them is a major problem. We think that the exponent for these extremal sets is the one attained in the upper bound.

Conjecture 1 *For any $r \geq 2$ and $2 \leq \ell_1 \leq \cdots \leq \ell_r$, we have*

$$F(n, \mathscr{L}^{(r)}_{\ell_1,\cdots,\ell_r}) \asymp n^{1-1/(\ell_1\cdots\ell_{r-1})}.$$

In the particular case of the $\mathscr{L}^{(3)}_{2,2,2}$-free sets, Theorem 2 gives the lower bound $F(n, \mathscr{L}^{(3)}_{2,2,2}) \gg n^{1-3/7-o(1)}$, however Katz, Krop and Maggioni [6] found a construction which gives

$$F(n, \mathscr{L}^{(3)}_{2,2,2}) \gg n^{1-1/3}. \tag{1}$$

We confirm this last lower bound with an alternative construction based upon a $\mathscr{L}^{(3)}_{2,2,2}$-free set in $\mathbb{Z}^3_{p-1}$.

Definition 3 *Given a finite abelian group G, we will denote by $F(G, \mathscr{L}^{(r)}_{\ell_1,\cdots,\ell_r})$ the largest size of a $\mathscr{L}^{(r)}_{\ell_1,\cdots,\ell_r}$-free set in G.*

Theorem 3 *For any prime $p \geq 2$ we have*

$$F(\mathbb{Z}^3_{p-1}, \mathscr{L}^{(3)}_{2,2,2}) \geq (p-3)^2.$$

The set we construct to prove Theorem 3 can be easily projected to the integers to prove (1), as it was done in [6]. In general we have

Proposition 1 *For any $r \geq 2$, and $2 \leq \ell_1 \leq \cdots \leq \ell_r$, and $k, n_1, \dots, n_k$ we have*

$$F(2^{k-1}n_1\cdots n_k, \mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}) \geq F(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}, \mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}).$$

To prove this proposition any sumset of prescribed size summands is characterized as a multiset satisfying certain conditions. Then a map is used between the two ambient groups, preserving the conditions satisfied by the multiset.

2 Related extremal problems in graphs and hypergraphs

The above-mentioned problems are related with extremal problems on graphs that are free of complete r-partite subgraphs. Given a graph $\mathscr{H}$, let $\mathrm{ex}(n, \mathscr{H})$ denote the maximum number of edges (or hyperedges) of a n vertices graph (or hypergraph) which does not contain $\mathscr{H}$ as a sub-graph (or sub-hypergraph). Estimating $\mathrm{ex}(n, \mathscr{H})$ is a major problem in extremal graph theory. An important case is when $\mathscr{H} = K_{\ell_1,\ell_2}$. It is known that

$$n^{2-\frac{\ell_1+\ell_2-2}{\ell_1\ell_2-1}} \ll \mathrm{ex}(n, K_{\ell_1,\ell_2}) \leq \frac{1}{2}(\ell_2-1)^{1/\ell_1} n^{2-\frac{1}{\ell_1}}(1+o(1)). \tag{2}$$

The upper bound was obtained by Kövari, Sós and Turán [7] and the lower bound can be easily obtained using the probabilistic method.

Definition 4 *Let $r \geq 2$ and $2 \leq \ell_1 \leq \cdots \leq \ell_r$ be integers. We write $K^{(r)}_{\ell_1,\dots,\ell_r}$ for the r-uniform hypergraph $(V,\mathscr{E})$ where $V = V_1 \cup \cdots \cup V_r$ with $|V_i| = \ell_i,\ i = 1,\dots,r$ and $\mathscr{E} = \{\{x_1,\dots,x_r\} :\ x_i \in V_i,\ i = 1,\dots,r\}$.*

We will say that the r-hypergraph $\mathscr{H}$ is $K^{(r)}_{\ell_1,\dots,\ell_r}$-free when $\mathscr{H}$ does not contain any r-uniform hypergraph $K^{(r)}_{\ell_1,\dots,\ell_r}$.

An easy probabilistic argument gives a lower bound which generalizes (2):

$$n^{r-\frac{\ell_1+\cdots+\ell_r-r}{\ell_1\cdots\ell_r-1}} \ll \mathrm{ex}(n;K^{(r)}_{\ell_1,\cdots,\ell_r}). \tag{3}$$

The upper bound was considered by Erdős in the case $\ell = \ell_1 = \cdots = \ell_r$. He proved [3, Theorem 1] that

$$ex(n,K^{(r)}_{\ell,\dots,\ell}) \ll n^{r-1/\ell^{r-1}}. \tag{4}$$

We refine the estimate (4) as follows.

Theorem 4

$$ex(n,K^{(r)}_{\ell_1,\dots,\ell_r}) \leq \frac{(\ell_r-1)^{1/\ell_1\cdots\ell_{r-1}}}{r!}\, n^{r-1/\ell_1\cdots\ell_{r-1}}(1+o(1)), \quad (n\to\infty). \tag{5}$$

The case $r = 2$ in Theorem 4 is the result (2) proved by Kövari, Sós and Turán [7]. It is believed that the upper bound in (5) is not far from the real value of $ex(n,K^{(r)}_{\ell_1,\dots,\ell_r})$.

Conjecture 2 *For any $r \geq 2$ and $2 \leq \ell_1 \leq \cdots \leq \ell_r$, we have*

$$ex(n,K^{(r)}_{\ell_1,\cdots,\ell_r}) \asymp n^{r-1/\ell_1\cdots\ell_{r-1}}.$$

The two exponents of n in Theorems 1 and 2 have the same flavour as the two exponents of n in (3) and in Theorem 4. This is a consequence of the following result which connects results on extremal problems in abelian groups with results on extremal problems in hypergraphs.

Proposition 2 *Let G be a finite abelian group with $|G| = n$. Then*

$$ex(n,K^{(r)}_{\ell_1,\dots,\ell_r}) \geq \binom{n}{r}\frac{F(G,\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r})}{n}.$$

The proof uses $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$-free sets in finite abelian groups to construct $K^{(r)}_{\ell_1,\dots,\ell_r}$-free hypergraphs.

3 $\mathscr{L}$-free infinite sequences

The problem on infinite $\mathscr{L}$-free sequences of positive integers is more difficult than the analogous finite problem, even in the simplest case of $\mathscr{L}_{2,2}$-free sets (Sidon sequences). Let $A(x) = |A\cap[1,x]|$ be the counting function of any sequence A. In the light of Conjecture 1 for the finite case and being optimistic one could believe in the existence of an infinite $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$-free sequence satisfying $A(x) \gg x^{1-1/(\ell_1\cdots\ell_{r-1})}$. Erdős [10] proved that it is not true for Sidon sequences, and Peng, Tesoro and Timmons [8] proved that neither for $\mathscr{L}_{\ell_1,\ell_2}$-free sequences this is true. We generalize these results for all $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$.

Theorem 5 *If A is an infinite $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$-free sequence then*

$$\liminf_{x\to\infty}\frac{A(x)}{x}(x\log x)^{1/(\ell_1\cdots\ell_{r-1})}\ll 1.$$

Hence a natural question is *whether or not for any $\varepsilon>0$ there exists a $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$-free sequence with*

$$A(x)\gg x^{1-1/(\ell_1\cdots\ell_{r-1})-\varepsilon}. \tag{6}$$

A positive answer to this question was conjectured by Erdős in the case of Sidon sequences, The greedy algorithm provides a Sidon sequence A with $A(x)\gg x^{1/3}$. This was the densest infinite Sidon sequences known during nearly 50 years. Ajtai, Komlós and Szemerédi [1] proved the existence of a Sidon sequence with $A(x)\gg(x\log x)^{1/3}$ and Ruzsa [9] proved the existence of a Sidon sequence with $A(x)\gg x^{\sqrt{2}-1+o(1)}$. The first author [2] constructed an explicit Sidon sequence with similar counting function.

To attain the exponent in (6) looks like a difficult problem. It even seems difficult to get a exponent greater than $1-\frac{\ell_1+\cdots+\ell_r-r}{\ell_1\cdots\ell_r-1}$, which is the exponent obtained in Theorem 2 for finite sets. The probabilistic method used in Theorem 2 might be adapted in order to prove that for every $\varepsilon>0$ there exist an infinite $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$-free sequence A satisfying

$$A(x)\gg x^{1-\gamma-\varepsilon},\ \text{with } \gamma=\frac{\ell_1+\cdots+\ell_r-r}{\ell_1\cdots\ell_r-1}.$$

We have not found a proof for the general case, however we have succeeded in two particular cases.

Theorem 6 *For any $\ell\ge 2$ and for any $\varepsilon>0$ there exists an infinite $\mathscr{L}_{2,\ell}$-free sequence with*

$$A(x)\gg x^{1-\frac{\ell}{2\ell-1}-\varepsilon}.$$

Note however that the constructions in [9] and [2] provide a greater exponent for $\ell=2$ and $\ell=3$.

Theorem 7 *For any $r\ge 2$ and for any $\varepsilon>0$ there exists an infinite $\mathscr{L}^{(r)}_{2,\dots,2}$-free sequence with*

$$A(x)\gg x^{1-\frac{r}{2^r-1}-\varepsilon}.$$

In a nutshell the strategy to obtain these two theorems is as follows. We first construct a dense random sequence S. We will say that X is an *obstruction* (for S) when $X\subset S$ is a sumset of the class $\mathscr{L}^{(r)}_{\ell_1,\dots,\ell_r}$. The sequence S is likely to have infinitely many obstructions. If we could proof that obstructions are few then we would be able to remove all of them by just removing few elements from S. In the cases of Hilbert cubes and $\mathscr{L}_{2,\ell}$ we have succeeded to obtain an upper bound for the number of obstructions which allows to complete the proofs of Theorems 7 and 6.

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