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An analog of Ferrero-Washtington theorem over the Carlitz-cyclotomic extension

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Ferrero-Washington theorem [3] is one of the most famous results on classical cyclotomic Iwasawa theory over $\mathbb{Q}$. In the talk we will briefly sketch how one deduces an analog of Ferrero-Washington theorem in the commutative "cyclotomic à la Carlitz" non-noetherian Iwasawa theory over $\mathbb{F}_q(T)$ (the rational fields over a finite field $\mathbb{F}_q$) from the Iwasawa Main Conjecture proved in the above setting in [1], the above result is also quoted as an application of the Iwasawa main conjecture in [1].

Let us first recall the classical result of Ferrero-Washington, but we follow an approach which is not the usual one (for the usual presentation we refer to [5, Chapter 7]).

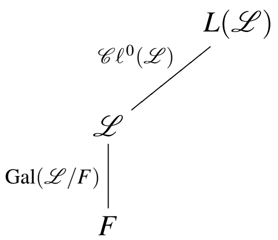
We are interested for the following module of class groups:
take F a global field (i.e. a finite extension of $\mathbb{F}_q(T)$ be the rational function field of characteristic p with constant field the finite field $\mathbb{F}_q$ or $\mathbb{Q}$) and E/F a finite extension

- $\mathscr{C}\ell^0(E)$ the p-part of the group of divisor classes of E of degree 0 (or p-part of the class group);
- $L(E)$ the maximal unramified abelian p-extension of E;
- $\mathscr{C}\ell^0(E) \simeq \mathrm{Gal}(L(E)/E)$ via the (canonical) Artin map.
- $\mathscr{C}\ell^0(E)$ is a $\mathbb{Z}_p[\mathrm{Gal}(E/F)]$-module, the action is provided by conjugation.

Same notations for infinite extensions $\mathscr{L}/F$ where

$$\mathscr{C}\ell^0(\mathscr{L}) := \varprojlim_{E} \mathscr{C}\ell^0(E)$$

(the limit is on the natural norm maps as E runs among the finite subextensions of $\mathscr{L}/F$) and becomes an $\mathbb{Z}_p[[Gal(\mathscr{L}/F)]] = \varprojlim_{F\subset E\mathscr{L}} \mathbb{Z}_p[Gal(E/F)]$-module.



$\mathscr{C}\ell^0(\mathscr{L})$ is a $\mathbb{Z}_p[[Gal(\mathscr{L}/F)]]$-module (the action is provided by conjugation)

Suppose that $Gal(\mathscr{L}/F)$ is abelian and E/F finite abelian extension with $Gal(\mathscr{L}/E) \cong \mathbb{Z}_p^d$ (and $\mathscr{L}/F$ ramified in a finite set of places) then by theorems of Iwasawa, Greenberg, and many others, we know that

$$\mathscr{C}\ell^0(\mathscr{L}) \text{ is a torsion finitely generated } \mathbb{Z}_p[[Gal(\mathscr{L}/F)]]-module.$$

We are interested in understanding such finitely generated torsion module and for particular reasons we will be interested in χ-components of the above modules where χ is a p-adic character of $Gal(E/F)$.

Let us now restrict to the initial case concerning Ferrero-Washington theorem.

Let us consider $\mathbb{Q}_n = \mathbb{Q}(\zeta_{p^n})$ where ζ_{p^n} is a primitive p^n-th root of unity, and consider what is called the cyclotomic extension $\mathbb{Q}_{cyc}/\mathbb{Q}$ with $\mathbb{Q}_{cyc} = \cup_{n\geq 1}\mathbb{Q}_n$, observe that $\mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \times \mathbb{Z}_p$, and denote Γ by $Gal(\mathbb{Q}_{cyc}/\mathbb{Q}_1) \cong \mathbb{Z}_p$ in the number field setting.

Consider a p-adic Dirichlet character $\chi : \mathrm{Gal}(\mathbb{Q}_1/\mathbb{Q}) \to \overline{\mathbb{Q}_p}^*$ and by $\mathscr{O}_\chi \subset \overline{\mathbb{Q}_p}$ we mean the ring extension of $\mathbb{Z}_p$ generated by the values of χ, where as usual $\mathbb{Q}_p$ are the p-adic numbers and $\mathbb{C}_p$ will denote the completion of $\overline{\mathbb{Q}_p}$.

For any $\mathbb{Z}_p$-module U which admits a continuous action of $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, the χ-part of U is the $\mathscr{O}_\chi[[\Gamma]]$-module obtained by:

$$U(\chi) := e_\chi U$$

where $e_\chi = \frac{1}{|\mathrm{Gal}(\mathbb{Q}_1/\mathbb{Q})|}\sum_{\delta\in Gal(\mathbb{Q}_1/\mathbb{Q})}\chi(\delta^{-1})\delta$.

In particular $\mathscr{C}\ell^0(\mathbb{Q}_{cyc})(\chi)$ is a $\mathscr{O}_\chi[[\Gamma]]$-module which is torsion and finitely generated.

Recall that we have a non-canonical isomorphism:

$$\sigma_\gamma : \mathscr{O}_\chi[[\Gamma]] \to \Lambda_\chi := \mathscr{O}_\chi[[T]]$$

mapping γ (a topological generator of Γ) to $1+T$, latter on we will fix this choice.

Denote by Λ_χ the Iwasawa module $\mathscr{O}_\chi[[\Gamma]]$.

Recall the following statements on the theory of finitely generated Λ_χ-modules:

1. M,N are Λ_χ-pseudo-isomorphic if exists a Λ_χ-homomorphism from M to N with finite kernel and cokernel. We write $M \sim N$ if they are Λ_χ-pseudo-null, which is an equivalence relation for Λ_χ-torsion modules.

2. M a Λ_χ-torsion, then
$$M \sim \oplus_{i=1}^r \Lambda_\chi/(h_i)$$
for some natural r where each $h_i \in \Lambda_\chi$. The invariant
$$(h_1\cdots h_r) = Char_{\Lambda_\chi}(M)$$
is named the characteristic ideal of the Λ_χ-module M.

   Recall that for any $\alpha \in \Lambda_\chi$, we can write
$$\alpha = \pi^\mu h(T)v(T)$$
where π is an uniformizer of $\mathscr{O}_\chi$, $h(T)$ a distinguished polynomial of degree λ (i.e. the reduction of $h(T)$ in $\Lambda_\chi/(p)$ is T^λ), and $v(T)$ a unit of Λ_χ.

   For $Char_{\Lambda_\chi}(M) = (\alpha)$ with α as above, the number $\mu \in \mathbb{N}$ is called the μ-invariant of M, and $\lambda = degree_T(h(T))$ is named the λ-invariant of M.

For any character $\chi = \omega^i$ (i even and nonzero and ω the Teichmüller character) associated to $\mathrm{Gal}(\mathbb{Q}_1/\mathbb{Q})$ Iwasawa defined p-adic analogues $L_p(s,\omega^i)$ of the Dirichlet L-series $L(s,\omega^i)$ with interpolation property:

$$L_p(1-m,\omega^i) = (1-\omega^{i-m}(p)p^{m-1})L(1-m,\omega^{i-m}) \ for\ m \geq 1$$

Moreover Iwasawa proved that there exist a power series $f(T,\omega^i) \in \mathbb{Z}_p[[T]] \simeq \Lambda_\chi$ such that $L_p(s,\omega^i) = f((1+p)^s - 1, \omega^i)$.

Conjecture 1 (Main Conjecture (MC)) *With i even and non-zero, let $\mathscr{C}\ell^0(\mathbb{Q}_{cyc})(\omega^i)$ be the ω^i-part of the Iwasawa module, then*

$$Ch_{\Lambda_\chi}(\mathscr{C}\ell^0(\mathbb{Q}_{cyc})(\omega^i)) = (f(T,\omega^{1-i}))\ .$$

Mazur-Wiles in [4] proved the above Main Conjecture (also in a more general result). Different proofs and many generalizations have been investigated since then.

By the above result of Mazur-Wiles we can formulate the result of Ferrero-Washington in [3] as follows:

Theorem 2 (Ferrero-Washington) *With the hypothesis of the previous theorem, the power series $f(T,\omega^{1-i})$ satisfies that, $f(T,\omega^{1-i}) \not\equiv 0 (mod\ p)$.*

The usual formulation of Ferrero-Washington theorem claims that the μ-invariant associated to the characteristic ideal of the module $\mathscr{C}\ell^0(\mathbb{Q}_{cyc})$ is zero as $\mathbb{Z}_p[[Gal(\mathbb{Q}_{cyc}/\mathbb{Q})]]$-torsion module.

Now let us consider the "cyclotomic Carlitz" extension.

Let $F := \mathbb{F}_q(\theta)$ be the rational function field of characteristic p and fix $\frac{1}{\theta}$ as the prime at ∞. Let Φ be the Carlitz module associated with $A := \mathbb{F}_q[\theta]$ and fix a prime $\mathfrak{p}$ of A. For any $n \geqslant 0$ let $\Phi[\mathfrak{p}^n]$ be the $\mathfrak{p}^n$-torsion of the Carlitz module and $F_n := F(\Phi[\mathfrak{p}^{n+1}])$. We define the $\mathfrak{p}$-*cyclotomic extension* of F as $\mathscr{F} := \cup F_n$: it is a Galois extension with $\mathrm{Gal}(\mathscr{F}/F_0) := \Gamma \simeq \mathbb{Z}_p^\infty$. Let $\mathscr{C}\ell_n$ be the p-part of the group of divisor classes of degree zero of F_n and put $\mathscr{C}\ell^0(\mathscr{F})$ as the inverse limit (on the natural norm maps) of the $\mathscr{C}\ell_n$. We study $\mathscr{C}\ell^0(\mathscr{F})$ as a module over the non-noetherian Iwasawa algebra $\mathbb{Z}_p[\mathrm{Gal}(F_0/F)][[\Gamma]]$.

Consider χ a character of $Gal(F_0/F)$ which has values on $W \cong \mathbb{Z}_p[\mu_{q^d-1}]$ we study next $\mathscr{C}\ell^0(\mathscr{F})(\chi)$ as $W[[\Gamma]]$-module.

We prove in [1]

Proposition 3 (Anglès-Bandini-B.-Longhi) *For χ non-trivial then the Iwasawa module $\mathscr{C}\ell^0\mathscr{F}(\chi)$ is a finitely generated torsion $W[[\Gamma]]$-module.*

In such Iwasawa algebras we do not have a natural characteristic element, also it is not known yet a structure theorem, but we can deal with Fitting ideals instead of Characteristic ideals (see [2] for an approach on defining some sort of pro-Characteristic ideal by use some filtration that may be useful in the future with the use of Waldhausen categories in the formulation of Iwasawa Main Conjectures).

Let now Λ_χ any Iwasawa algebra, for example $W[[\Gamma]]$.

Definition 4 *Let Z be a finitely generated Λ_χ-module and let*

$$\Lambda_\chi^a \to^\psi \Lambda_\chi^b \twoheadrightarrow Z$$

be a presentation, where the map ψ can be represented by an $a \times b$-matrix Φ_Z with entries in Λ_χ.

In this setting, the Fitting ideal of Z is the ideal generated by all the determinants of the $b \times b$-minors of Φ_Z if $a \geq b$ and otherwise is the zero ideal. We denote this ideal by $Fitt_{\Lambda_\chi}(Z)$.

There are a lot of results concerning Fitting ideals (which usually is not a principal ideal) versus Characteristic ideals, in particular if the module has projective dimension one both concepts coincide and a lot of technical results.

The main result concerning IMC in [1] reads as follows:

Theorem 5 (Anglès-Bandini-B.-Longhi) *For χ a non-trivial p-adic character of $Gal(F_0/F)$ we have:*

$$\mathrm{Fitt}_{W[[\Gamma]]}(\mathscr{C}\ell^0(\mathscr{F})(\chi)) = (\lim_{\overleftarrow{n}} \Theta_n^{\#}(1,\chi)) =: (\Theta_\infty^{\#}(1,\chi))$$

with

$$\Theta_n^{\#}(1,\chi) := \begin{cases} \Theta_n(1,\chi) & \text{if } \chi \text{ is odd} \\ \frac{\Theta_n(X,\chi)}{1-X}|_{X=1} & \text{if } \chi \text{is even} \end{cases} .$$

where $\Theta_n(X,\chi)$ are χ-component of the usual Stickelberger series defined of $G_n = \mathrm{Gal}(F_n/F)$ by

$$\Theta_n(X) := \Theta_{n,S=\{\mathfrak{p},\infty\}}(X) = \prod_{v\notin S}(1-\mathrm{Fr}_v^{-1}X^{d_v})^{-1} \in \mathbb{Z}[G_n][[X]] ,$$

with $d_v := \deg(v)$ and Fr_v (lifting of) the Frobenius of v in $Gal(F_n/F)$. Here χ is even if $\chi(\mathbb{F}_q^*) = 1$ and odd otherwise.

As a consequence of the above theorem on Iwasawa main conjecture we obtain in [1] the following:

Theorem 6 (An analog of Ferrero-Washington in [1]) *For χ a non-trivial character of $Gal(F_0/F)$ we have:*

$$\Theta_\infty^{\#}(1,\chi) \not\equiv 0 (mod\ p).$$

In the talk we will explain briefly the ingredients to deduce the above analog of Ferrero-Washington theorem from the core theorem 5 concerning Iwasawa Main Conjecture for the Carlitz cyclotomic extension.

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