LIBRO DE RESÚMENES

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Presentación

Del 7 al 9 de Julio de 2010 se celebró en el CIEM (Centro Internacional de Encuentros Matemáticos, Castro Urdiales) la séptima edición de las Jornadas de Matemática Discreta y Algorítmica. Se trata de una reunión bianual nacida en Barcelona en 1998 y que ha pasado posteriormente por Palma de Mallorca (2000), Sevilla (2002), Cercedilla (2004), Soria (2006), y Lérida (2008).

En total, forman estas actas tres conferencias invitadas y 48 comunicaciones, 33 de las cuales se presentaron de forma oral y las otras 15 en forma de póster, por falta de tiempo en el programa. Eso sí, desde los Comités Científicos y Organizador decidimos que, para dar un formato más manejable a las actas, se recogerían en papel sólo unos pequeños resúmenes de las comunicaciones, con un máximo de dos páginas, invitándose a los autores a enviar también una versión más completa que se distribuye sólo en forma de CD con ISBN, así como en la web del congreso (www.jmda2010.unican.es).

La celebración de este evento, y la constatación de la madurez a la que podemos decir que ha llegado la disciplina en nuestro país, es un motivo de satisfacción para los organizadores, tanto el Comité Organizador, formado por doce miembros todos de la Universidad de Cantabria, como para el Comité Científico, formado por ocho investigadores de toda España, aunque con singular presencia catalana que refleja la especial buena salud que la matemática discreta goza en aquella comunidad (presencia que también se nota en el número de comunicaciones).

En el apartado de agradecimientos, además de los miembros de los comités, debemos mencionar al Proyecto i-MATH, al Ayuntamiento de Castro Urdiales, al Ministerio de Ciencia e Innovación, a la Universidad de Cantabria, a la Red Temática MATSI y a la Real Sociedad Matemática Española.

Y, sobre todo, a todos los participantes y ponentes de las Jornadas, que son los que en última instancia hacen de éstas unos encuentros interesantes, fructíferos, y a la vez intensos y distendidos.

Daniel Sadornil Renedo, presidente del Comité Organizador

Francisco Santos Leal, presidente del Comité Científico

Índice general

CONFERENCIAS INVITADAS

Equipartitioning measures and functions by a k-fan

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Abstract. We prove that, given a planar convex set K of positive area, there exist three pairwise internally disjoint convex sets whose union is K such that they have equal area and equal perimeter. Several other questions of similar type are also considered.

Key words: Equipartitioning measures, k-fans, equivariant topology

1 Introduction

More than ten years ago Kaneko and Kano [KK] raised the following question. Given kr red points and kb blue points in the plane (in general position, say), is there a convex k-partition of the plane into pieces $P_1, \dots, P_k$ such that each piece contains exactly r red and b blue points? Here $P_1, \dots, P_k$ form a convex k-partition of $\mathbb{R}^2$ if each P_i is convex, they are internally disjoint, and their union is $\mathbb{R}^2$. Note that the Borsuk Ulam theorem immediately implies a positive answer to this question when k is a power of two. Indeed, there is a line ℓ splitting $\mathbb{R}^2$ into two halfplanes σ_1 and σ_2 such that both halfplanes contain exactly half the red and half the blue points, and one can repeat this argument in the halfpanes σ_1 and σ_2, etc.

The question of Kaneko and Kano proved to be fertile and gave rise to some neat results and several further problems. The original question was solved, affirmatively and independently, by Bespanyatnikh et al. [BKS] and by Sakai [Sak]. They showed a little more generally that, given an integer $k \geq 2$ and two absolutely continuous probability measures μ_1 and μ_2 in the plane, there is a convex k-partition, $P_1, \dots, P_k$ of the plane with $\mu_i(P_j) = \frac{1}{k}$ for all $i = 1, 2$ and

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$j = 1, \dots, k$. This result follows from another one about 3-fans and measures. A point x in the plane and three halflines, ℓ_1, ℓ_2, ℓ_3, starting from x form a 3-fan. The halflines are in anticlockwise order around x. They determine three angular sectors $\sigma_1, \sigma_2, \sigma_3$ with σ_i between ℓ_i and ℓ_{i+1}. The 3-fan is *convex* if each σ_i is convex. A k-fan is defined analogously. The positive answer for the Kaneko-Kano question follows from the following statement.

Theorem 1. *Assume μ_1 and μ_2 are absolutely continuous (with respect to the Lebesgue measure) Borel measures on $\mathbb{R}^2$ with $\mu_1(\mathbb{R}^2) = \mu_2(\mathbb{R}^2) = k$, where $k \geq 2$ is an integer. Then either*

- *there are positive integers k_1, k_2 with $k = k_1 + k_2$ and a line ℓ splitting $\mathbb{R}^2$ into two halfplanes σ_1 and σ_2 such that $\mu_i(\sigma_j) = k_j$ for every i and j,*
- *or there are positive integers k_1, k_2, k_3 with $k = k_1 + k_2 + k_3$ and a convex 3-fan such that $\mu_i(\sigma_j) = k_j$ for every i and j.*

The proof of this result is similar to, but more involved than, that of Sperner lemma. Now the following questions emerge naturally. Assume $\alpha_1, \dots, \alpha_k$ are positive numbers whose sum equals 1. Given (nice) probability measures $\mu_1, \dots, \mu_m$ on $\mathbb{R}^2$, is there a (convex) k-fan such that $\mu_i(\sigma_j) = \alpha_j$ for every i and j?

It is more convenient to lift the measures and the k-fans from $\mathbb{R}^2$ to the 2-sphere S^2 mainly because S^2 is compact. So let S^2 be the unit sphere of $\mathbb{R}^3$ and let $\mathbb{R}^2$ be embedded in $\mathbb{R}^3$ as the horizontal plane tangent to S^2 (at the South Pole). Denote by π the central projection from the lower hemisphere to the embedded $\mathbb{R}^2$. Clearly, π^{-1} lifts any Borel measure on $\mathbb{R}^2$ to a Borel measure on the lower hemisphere of S^2. A k-fan in $\mathbb{R}^2$ is lifted to a k-fan in S^2 in a natural way: a spherical k-fan is a point $x \in S^2$ and k great half circles $\ell_1, \ell_2, \dots, \ell_k$ starting at x (and ending at $-x$) that are ordered anticlockwise when viewed from x. The angular sector between ℓ_i and ℓ_{i+1} is σ_i. It is clear that a spherical k-fan is projected by π to a k-fan in $\mathbb{R}^2$, and conversely, a k-fan in $\mathbb{R}^2$ is mapped by π^{-1} to a spherical k-fan on S^2. A spherical k-fan is convex if the angle of each sector is at most π. It is also evident that a spherical k-fan is convex if and only if the corresponding planar k-fan is convex. The following theorem summarizes the known results.

Theorem 2. *Let $m \geq 2$, $k \geq 2$, and let $\alpha = (\alpha_1, \dots, \alpha_k)$ have no zero component.*

(i) (a) For any $k \geq 2$ and any α, there are 4 measures that cannot be simultaneously α-partitioned by a k-fan.
(b) For any $k \geq 3$ and any α, there are 3 measures that cannot be simultaneously α-partitioned by a k-fan.
(c) For any $k \geq 5$ and any α, there are 2 measures that cannot be simultaneously α-partitioned by a k-fan.

(d) For $k = 4$ and any α, there are 2 measures that cannot be simultaneously α-partitioned by a convex 4-fan.

(ii) (a) Any 2 measures can be simultaneously α-partitioned by a 2-fan, for all α. The center of the 2-fan can be prescribed arbitrarily.

(b) Any 3 measures can be simultaneously α-partitioned by a 2-fan for $\alpha = (\frac{1}{2}, \frac{1}{2})$ and for $\alpha = (\frac{2}{3}, \frac{1}{3})$.

(c) Any 2 measures can be simultaneously α-partitioned by a 3-fan, when $\alpha = (\beta, \beta, 1-2\beta)$ and when $\alpha = (\beta, 1/2, 1/2-\beta)$, $\beta \in (0, 1/2)$.

(d) Any 2 measures can be simultaneously α-partitioned by a 4-fan for $\alpha = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and for $\alpha = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

Statement (c) in part (ii) above is due to Blagojević and Dmitijević [BD], while the others come from Bárány and Matoušek [BM1] and [BM2]. What is striking here is the lack of positive statements or counterexamples concerning (d) in part (ii). The reason for this is that the only possible line of proof, at least at the moment, is by using (equivariant) algebraic topology. I'm going to explain how this can be done in the case of equipartitioning 3-fans, that is, when $\alpha = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $m = 2$.

We need a definition. A measure on S^2 is called *nice* if it is a probability measure that has a positive and continuous density function on S^2. In what follows all measures are supposed to be nice.

2 Equivariant topology

Write $V = \{(x, y) \in S^2 \times S^2 \, x \perp y\}$; V is the Stiefel manifold of all orthogonal 2-frames in $\mathbb{R}^3$, which is homeomorphic to $SO(3)$. We use the facts that V is orientable and its fundamental group is Z_2. To every $(x, y) \in V$ we assign the 3-fan $(x; \ell_1, \ell_2, \ell_3)$ as follows: y is the midpoint of the half great circle ℓ_1 whose endpoints are x and $-x$, and ℓ_2, ℓ_3 are defined by the condition $\mu(\sigma_i) = \frac{1}{3}$ for all i. As μ is nice, the half great circles ℓ_i and the sectors σ_i are determined uniquely. Thus the mapping $(x, y) \to (x; \ell_1, \ell_2, \ell_3)$ is well-defined and we can simply write ℓ_i or σ_i for $\ell_i(x, y)$ and $\sigma_i(x, y)$.

Next, let y^i be the midpoint of the great half circle ℓ_i. So $y = y^1$. Define $\omega : V \to V$ via $\omega(x, y) = \omega(x, y^1) = (x, y^2)$. Thus $\omega^2(x, y) = (x, y^3)$. It is clear that $\omega^3 = \mathrm{id}_V$ so the cyclic groups Z_3 acts on V and ω is the action of its generator. Further, ω has no fixed point and is a $V \to V$ homeomorphism that keeps the orientation of V since $\omega^3 = \mathrm{id}$.

We wish to show the existence of a 3-fan equipartitioning μ_1 and μ_2. Define $f : V \to \mathbb{R}^3$ by $f = (\mu_2(\sigma_1), \mu_2(\sigma_2), \mu_2(\sigma_3)) \in \mathbb{R}^3$. Observe that Z_3 acts on $\mathbb{R}^3$ by shifting the coordinates. That is, with $\omega = \omega_{\mathbb{R}^3}$, $\omega(t_1, t_2, t_3) = (t_2, t_3, t_1)$. It is clear that f is a Z_3-equivariant map, that is, $f \circ \omega = \omega \circ f$. Here the first ω acts on V, the second acts on $\mathbb{R}^3$ and is $\omega_{\mathbb{R}^3}$.

Proposition 1. *Under the above conditions there is* $(x, y) \in V$ *such that* $f(\sigma_1) = f(\sigma_2) = f(\sigma_3)$.

This is the usual *phase space/test map* method, with V being the phase space, f the test map, and we want to show that the test map cannot avoid the diagonal $\triangle = \{(t, t, t) \in \mathbb{R}^3\}$.

Proof. We assume the contrary which means that f avoids the diagonal. This gives rise to a chain of maps

$$V \to \mathbb{R}^3 \to \triangle^\perp \to S^1$$

where the first arrow is f, the second is the orthogonal projection onto $\triangle^\perp$ (the orthogonal complement of $\triangle$), and the last arrow maps $v \in \triangle^\perp$, $(v \neq 0)$ to $v/|v| \in S^1$ (the unit circle in $\triangle^\perp$). Let g denote the composition map $V \to S^1$. On this S^1, $\omega = \omega_{\mathbb{R}^3}$ acts as a rotation by $2\pi/3$. It is easy to check that g is a Z_3-equivariant map: $g \circ \omega = \omega \circ g$.

Consider the cycle $C = \{(e_3, y) \in V : \text{ all } y\}$, this is an S^1, so $g|_C : C \to S^1$ is an $S^1 \to S^1$ Z_3-map. By a theorem of Krasnoselskii and Zabreiko [KZ], the degree of $g|_C$ is $1 \mod 3$.

Next, take two copies of C and denote it by $2C$. It follows that the degree of $g|_{2C}$ is $2 \mod 3$. The fundamental group of V is Z_2, and so $2C$ is homotopic to 0 implying that the degree of $g|_{2C}$ is 0. A contradiction. □

The key step in this proof is the reduction of the geometric problem to the non-existence of a $V \to S^1$ Z_3-equivariant map. Such a proof cannot work if there exists such an equivariant map, and this is the case with many α in part (ii), (c) and (d) of Theorem 2, as shown by Blagojević, Živaljević and S. T. Vrećica [BŽV].

3 Equal perimeter

A similar and annoyingly resistant question has been recently asked by R. Nandakumar and N. Ramana Rao [NR]. Given a convex set K of positive area and an integer $k \geq 2$, does there exist a convex k-partition of $\mathbb{R}^2$ such that all parts $K \cap P_i$ have equal area and equal perimeter. For $k = 2$ the answer is yes, quite trivially, but is not known when $k > 3$. The answer is yes, again, when $k = 3$ as shown by Bárány, Blagojević and Szűcs in [BBS]. We state the theorem in this case slightly more generally and for S^2. It is easy to see that the S^2 case implies that of $\mathbb{R}^2$.

Theorem 3. *Assume* μ *is a nice probability measure on* S^2*, and* h *is a continuous function defined on the sectors in* S^2*. Then there is a convex 3-fan with* $\mu(\sigma_i) = \frac{1}{3}$ *for* $i = 1, 2, 3$ *and* $h(\sigma_1) = h(\sigma_2) = h(\sigma_3)$.

The case $k = 3$ of the Nandakumar, Ramana Rao conjecture follows from this by taking $f(\sigma)$ to be the perimeter of $K \cap \sigma$. Theorem 3 implies the existence of a convex 3-partition of $K \subset \mathbb{R}^2$ where the pieces have equal diameter, or equal width, etc. We mention that every convex 3-partition of $\mathbb{R}^2$ come from a convex 3-fan. One of the difficulties in the case of $k > 3$ is the lack of nice or natural description of convex k-partitions.

The proof in the previous section goes through without any significant change when f is equal to $(h(\sigma_1), h(\sigma_2), h(\sigma_3))$, the problem is that it does not guarantee that the resulting 3-fan is convex. One has to find a subset, V^{conv}, of V containing only convex 3-fans and establish its topological properties, in order to extend the previous proof. What is interesting here is that V^{conv} depends on the measure μ, so the phase space is not given a priori but has to be constructed from the data. Details can be found in [BBS].

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The computation of rational solutions of polynomial systems over a finite field *

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Abstract. We describe a family of probabilistic algorithms which compute a rational solution of a system of polynomial equations defined over a finite field $\mathbb{F}_q$. We express their cost in terms of numerical invariants associated to the geometry of the solution set and discuss the cases where the answer is not satisfactory.

Key words: Finite fields, rational points, algorithms, probability, irreducibility.

1 Introduction

Let $\mathbb{F}_q$ be the finite field of q elements and let $\overline{\mathbb{F}}_q$ be its algebraic closure. For a subfield $\mathbb{K}$ of $\overline{\mathbb{F}}_q$ extending $\mathbb{F}_q$, we shall denote by $\mathbb{K}[X_1, \ldots, X_n]$ the ring of polynomials in n indeterminates $X_1, \ldots, X_n$ and coefficients in $\mathbb{K}$.

Let be given polynomials $f_1, \ldots, f_m \in \mathbb{F}_q[X_1, \ldots, X_n]$. In this article we consider the problem of computing a q–rational solution of the system $f_1 = 0, \ldots, f_m = 0$, i.e., a point $\mathbf{x} \in \mathbb{F}_q^n$ such that $f_i(\mathbf{x}) = 0$ holds for $1 \leq i \leq m$.

Algorithms for finding rational solutions of polynomial systems over finite fields are usually based on rewriting techniques. Such algorithms use the standard dense complexity model, which typically requires an exponential number of operations. Families of algorithms based on rewriting techniques are Gröbner basis methods, XL methods and the Zhuang–Zi algorithm.

In a second family of randomized algorithms the geometry of the solution set of the system under consideration plays a critical role. As a first example of this family we may mention the algorithm of [6] for generating uniformly a random rational solution of a bivariate equation. Another algorithm in this direction is that of [4], which solves an *arbitrary* multivariate system by means of a perturbation of the original system and a subsequent path–following method.

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Finally, we mention the algorithm of [1], which solves "regular" systems having an absolutely irreducible solution set with a quadratic cost in the Bézout number of the system (the product of the degrees of the defining polynomials).

In what follows we shall present the principles underlying the design of a family of algorithms which compute a rational solution of a polynomial system over a finite field using the geometric approach of [6] and [1]. We show that the algorithms behave well when the solution set is absolutely irreducible and that the worst case arises when the solution set is relatively irreducible. We also comment on the frequency with which these two cases occur.

2 Plane curves

For a given $f \in \mathbb{F}_q[X,Y]$ of degree $d > 1$, we describe an algorithm for computing a q–rational solution of the equation $f = 0$. We shall express the question in terms of the *plane curve* $C := V(f) := \{(x,y) \in \overline{\mathbb{F}}_q^2 : f(x,y) = 0\}$.

Suppose that $q > 16d^4$ holds and f is absolutely irreducible. Following [6], we shall search for a q–rational zero of f in a suitable "vertical strip" $S_a := \{a\} \times \mathbb{F}_q$. Define $C_a(\mathbb{F}_q) := \{b \in \mathbb{F}_q : f(a,b) = 0\}$ for any $a \in \mathbb{F}_q$. From the well–known Weil estimate one deduces that, after at most d random choices, we will find $a \in \mathbb{F}_q$ such that $C_a(\mathbb{F}_q) \neq \emptyset$ with probability at least $1 - q^{-1/2}d^2 > 3/4$. This suggests making random choices in $\mathbb{F}_q$ until an element $a \in \mathbb{F}_q$ with $C_a(\mathbb{F}_q) \neq \emptyset$ is obtained. From this value we compute a zero $b \in \mathbb{F}_q$ of $f(a,Y)$. This can be done with $O(d\mathcal{U}(d)\log(dq))$ operations in $\mathbb{F}_q$.

Suppose now that f is not absolutely irreducible and let $f = f_1 \cdots f_r$ be its irreducible factorization in $\mathbb{F}_q[X,Y]$. If there exists an absolutely irreducible factor f_j, then we may proceed as above. On the other hand, if all the irreducible factors f_j are *relatively* $\mathbb{F}_q$*–irreducible*, i.e., irreducible in $\mathbb{F}_q[X,Y]$ but not absolutely irreducible, then every q–rational zero of f is a *singular point* of f ([4, Lemma 2.1]). It follows that the q–rational zeros of f are common zeros of f and its first partial derivatives. Using resultants, a q–rational zero of f can be computed with $O(d\mathcal{U}(d^2)\log(dq))$ operations in $\mathbb{F}_q$.

Combining the costs of both strategies with estimates on the probability that a random polynomial of $\mathbb{F}_q[X,Y]$ of degree $d > 1$ is reducible or relativamente $\mathbb{F}_q$–irreducible ([5]), we obtain the following result.

Theorem 1. *Let $f \in \mathbb{F}_q[X,Y]$ be a square–free polynomial of degree $d > 1$ and let $f = f_1 \cdots f_r$ be its irreducible factorization in $\mathbb{F}_q[X,Y]$. Assume that $q > 16d^4$ holds. Then a q–rational zero of f can be computed with an expected number of $O(d\mathcal{U}(d)\log(dq))$ operations in $\mathbb{F}_q$.*

3 Hypersurfaces

In this section we describe an algorithm for computing a q–rational zero of an n–variate polynomial $f \in \mathbb{F}_q[X_1, \ldots, X_n]$. For this purpose, we shall consider

the *hypersurface (defined over $\mathbb{F}_q$)* $H := V(f) := \{\mathbf{x} \in \overline{\mathbb{F}}_q^n : f(\mathbf{x}) = 0\}$. The *degree* of H is the minimal degree of a polynomial $f \in \overline{\mathbb{F}}_q[X_1, \ldots, X_n]$ defining H. Absolute and relative irreducibility are defined as in Section 2.

Suppose that $q > 16d^4$ holds and H is absolutely irreducible. We search for a q–rational point in a *plane section* $H \cap L$ of H, where $L \subset \overline{\mathbb{F}}_q^n$ is *plane* defined over $\mathbb{F}_q$. For this purpose, it is essential that the curve $H \cap L$ is absolutely irreducible. We shall use the following *effective Bertini theorem*: if $d := \deg H > 1$, then the probability that $H \cap L$ is not absolutely irreducible for a randomly chosen plane L defined over $\mathbb{F}_q$ is at most $2d^4/q \leq 1/8$.

We choose L at random and compute a q–rational point of $H \cap L$, which is then lifted to a q–rational point of H using the parametrization of L. We state the cost of this procedure in the next result.

Theorem 2. *Let $f \in \mathbb{F}_q[X_1, \ldots, X_n]$ be absolutely irreducible of degree $d > 1$ and let $H := V(f) \subset \overline{\mathbb{F}}_q^n$. For $q > 16d^4$, a q–rational point of H can be computed with an expected number of $O(\mathcal{L}\, d^2 + d\,\mathcal{U}(d)\log(dq))$ operations in $\mathbb{F}_q$, where $\mathcal{L}$ is the number of operations in $\mathbb{F}_q$ required to evaluate f.*

Suppose now that f is square–free and let $f = f_1 \cdots f_r$ be its irreducible factorization in $\mathbb{F}_q[X_1, \ldots, X_n]$. If there is an absolutely irreducible factor f_j, then we apply the previous algorithm. If all the f_j are relatively $\mathbb{F}_q$–irreducible, then every q–rational zero of f is a *singular point* of f ([4, Lemma 2.1]).

Let $g := \partial f/\partial X_j$ be a nonzero first partial derivative of f. Then the resultant $h_1 := \mathrm{Res}_{X_1}(f, g) \in \mathbb{F}_q[X_2, \ldots, X_n]$ is a nonzero polynomial of degree $O(d^2)$ vanishing on $H(\mathbb{F}_q)$. If h_1 has an absolutely irreducible factor defined over $\mathbb{F}_q$, then we can find a q–rational zero $\mathbf{x} \in \mathbb{F}_q^{n-1}$ of h_1, *thus reducing the number of variables* involved. If not, we consider a further polynomial in $\mathbb{F}_q[X_3, \ldots, X_n]$ of degree $O(d^4)$. In this way we eventually arrive at a polynomial of $\mathbb{F}_q[X_n]$ of degree $O(d^{2^n})$, having $d^{2^{O(n)}} \log^{O(1)} q$ operations in $\mathbb{F}_q$.

Estimates in [7] imply that the probability that this worst case occurs may be very low. Furthermore, the average–case complexity of the algorithm above might be essentially that of the absolutely irreducible case.

4 Varieties

For given $f_1, \ldots, f_m \in \mathbb{F}_q[X_1, \ldots, X_n]$, we now discuss the problem of finding a q–rational solution of the system $f_1 = 0, \ldots, f_m = 0$. Therefore, we consider the *(algebraic) variety* $V := V(f_1, \ldots, f_m) := \{\mathbf{x} \in \overline{\mathbb{F}}_q^n : f_i(\mathbf{x}) = 0\ (1 \leq i \leq m)\}$.

Observe that $V \subset \overline{\mathbb{F}}_q^n$ is *defined over* $\mathbb{F}_q$ and thus can be expressed as a finite union of $\mathbb{F}_q$*–irreducibles* varieties, which are called the $\mathbb{F}_q$*–irreducible components* of V. For $\overline{\mathbb{F}}_q$–definability we speak of *absolutely irreducible* varieties and the *absolutely irreducibles* components of V.

Let $d := \max_{1 \leq i \leq m} \deg(f_i)$. Suppose that $q \geq 8n^2 d\delta^4$ holds and V is absolutely irreducible of dimension $r > 0$ and degree δ. Then it can be proved

that there exists a linear "birational" projection $\pi : V \to \overline{\mathbb{F}}_q^{r+1}$ defined over $\mathbb{F}_q$ such that $\pi(V)$ is an hypersurface and the inverse function ϕ of π is well–defined in an open dense subset of $\pi(V)$.

In order to compute a q–rational point of V, we first find such a projection π and compute a polynomial $h \in \mathbb{F}_q[Y_1, \ldots, Y_{r+1}]$ of minimal degree defining $H := \pi(V)$. This can be done with an algorithm of [1]. Then we compute $\mathbf{y} \in H(\mathbb{F}_q)$ and $\mathbf{x} := \phi(\mathbf{y}) \in V(\mathbb{F}_q)$. We have thus the following result.

Theorem 3. *Let $f_1, \ldots, f_m \in \mathbb{F}_q[X_1, \ldots, X_n]$ be polynomials with $\deg(f_i) \leq d$ for $1 \leq i \leq m$ defining an absolutely irreducible variety $V := V(f_1, \ldots, f_m) \subset \overline{\mathbb{F}}_q^n$ of dimension $r > 0$ and degree δ. For $q \geq 8n^2 d\delta^4$, a q–rational point of V can be computed with an expected number of $O(n^5 \mathcal{U}(D)^2 \log q)$ operations in $\mathbb{F}_q$ (up to logarithmic terms), where $\mathcal{L}$ is the number of operations in $\mathbb{F}_q$ required to evaluate $f_1, \ldots, f_m$ and $D := \prod_{i=1}^m \deg(f_i)$.*

Similar results are obtained if V has an absolutely irreducible component defined over $\mathbb{F}_q$. On the other hand, significant problems arise when none of the $\mathbb{F}_q$-irreducible components of V is absolutely irreducible.

In [4], an algorithm which finds a q–rational solution of an *arbitrary* polynomial system using a geometric approach similar to the one above is described. The algorithm performs $d^{n^{O(n)}}(m \log q)^{O(1)}$ operations in $\mathbb{F}_q$, where $d := \max_{1 \leq i \leq m} \deg f_i$. It is not clear how to avoid this exponential behavior.

On the other hand, this worst case is likely to occur with a very low frequency. In this direction, in [3] it is shown that the probability that a projective curve defined over $\mathbb{F}_q$ is relatively reducible tends to zero as q tends to infinity.

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Counting graphs on surfaces

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Abstract. We show that the number of labelled graphs of genus g with n vertices grows asymptotically like

$$c^{(g)} n^{5(g-1)/2-1} \gamma^n n!$$

where $c^{(g)} > 0$, and $\gamma \approx 27.23$ is the exponential growth rate of planar graphs. This generalizes the known results for the planar case $g = 0$. We also discuss briefly properties of random graphs of genus g. Related results are discussed at the end of the paper.

Most of this extended abstract is based on joint work with Guillaume Chapuy, Éric Fusy, Omer Giménez and Bojan Mohar; see reference [6] in the bibliography.

Key words: Asymptotic enumeration. Graphs on surfaces. Random graphs.

1 Introduction and statement of main results

It has been shown by Giménez and Noy [16] that the number of planar graphs with n labelled vertices grows asymptotically as

$$c \cdot n^{-7/2} \gamma^n n!$$

where $c > 0$ and $\gamma \approx 27.23$ are well defined analytic constants. Since planar graphs are precisely those that can be embedded in the sphere, it is natural to ask about the number of graphs that can be embedded in a given surface.

In what follows, graphs are simple and labelled with $V = \{1, 2, \ldots, n\}$, so that isomorphic graphs are considered different unless they have exactly the same edges. Let $\mathbb{S}_g$ be the orientable surface of genus g, that is, a sphere with g handles, and let $a_n^{(g)}$ be the number of graphs with n vertices embeddable in $\mathbb{S}_g$. A first approximation to the magnitude of these numbers was given by McDiarmid [19], who showed that

$$\lim_{n\to\infty} \left(\frac{a_n^{(g)}}{n!} \right)^{1/n} = \gamma.$$

This establishes the *exponential growth* of the $a_n^{(g)}$, which is the same as for planar graphs and does not depend on the genus.

We provide a considerable refinement and obtain a sharp estimate, showing how the genus comes into play in the *subexponential growth*. In the next statements, γ is the exponential growth rate of planar graphs, and γ_μ is the exponential growth rate of planar graphs with n vertices and $\lfloor \mu n \rfloor$ edges. Both γ and the function γ_μ are determined analytically in [16]. All the results in this and the next section are proved in [6].

Theorem 1. *For $g \geq 0$, the number $a_n^{(g)}$ of graphs with n vertices that can be embedded in the orientable surface $\mathbb{S}_g$ of genus g satisfies*

$$a_n^{(g)} \sim c^{(g)} n^{5(g-1)/2-1} \gamma^n n! \tag{1}$$

where $c^{(g)}$ is a positive constant and γ is as before.

For $\mu \in (1,3)$, the number $a_{n,m}^{(g)}$ of graphs with n vertices and $m = \lfloor \mu n \rfloor$ edges that can be embedded in $\mathbb{S}_g$ satisfies

$$a_{n,m}^{(g)} \sim c_\mu^{(g)} n^{5g/2-4} (\gamma_\mu)^n n! \quad \textit{when } n \to \infty,$$

where $c_\mu^{(g)}$ is a positive constant and γ_μ is as before.

We also prove an analogous result for non-orientable surfaces. Let $\mathbb{N}_h$ be the non-orientable surface of genus h, that is, a sphere with h crosscaps.

Theorem 2. *For $h \geq 1$, the number $b_n^{(h)}$ of graphs with n vertices that can be embedded in the non-orientable surface $\mathbb{N}_h$ of genus h satisfies*

$$b_n^{(h)} \sim \tilde{c}^{(h)} n^{5(h-2)/4-1} \gamma^n n!$$

where $\tilde{c}^{(h)}$ is a positive constant and γ is as before.

For $\mu \in (1,3)$, the number $b_{n,m}^{(h)}$ of graphs with n vertices and $m = \lfloor \mu n \rfloor$ edges that can be embedded in $\mathbb{N}_h$ satisfies

$$b_{n,m}^{(h)} \sim \tilde{c}_\mu^{(h)} n^{5h/4-4} (\gamma_\mu)^n n! \quad \textit{when } n \to \infty,$$

where $\tilde{c}_\mu^{(h)}$ is a positive constant and γ_μ is as before.

In theory, the constants $c^{(g)}$ can be computed via non-linear recursions. Indeed, our computations relate $c^{(g)}$ to the asymptotic number of maps embedded on the surface $\mathbb{S}_g$, and weighted by their number of vertices, which are shown to obey such recursions in [3]. However, these recursions are so intricate that in practice it is not easy to compute even the first few of these numbers.

There are three main ingredients in our proof. The first one is the theory of map enumeration, started by Tutte in his pioneering work on planar maps [24], and extended later by Arquès, Bender, Canfield, Gao, Richmond, Wormald,

and others to arbitrary surfaces. Our main references in this context are [2], [3] and [4]. In particular, Bender and Canfield [2] showed that the number of rooted maps with n edges embeddable in $\mathbb{S}_g$ grows asymptotically as

$$t_g n^{5(g-1)/2} 12^n,$$

for some constant $t_g > 0$. If we compare it with Estimate (1) in Theorem 1, we see that they are very similar. This is no coincidence, since our counting of graphs relies in a fundamental way on the counting of maps (the extra factor of n in the estimate occurs because maps are rooted, and the absence of a factorial term is because they are unlabelled). The exponent $5(g-1)/2$ appears as a result of computations with generating functions, and for several years it has been a tantalizing problem to find a combinatorial explanation for it. This been recently achieved by Chapuy, based on encoding maps through well-labelled trees and the operation of gluing vertices in a map [5].

The second ingredient is topological graph theory, in particular the concept of face-width, which measures in some sense the local planarity of an embedding of a graph in a surface. More precisely, the face-width of an embedding of a graph G in a surface S is the minimum number of intersections of G with a non-contractible curve in S; equivalently, it is the minimum number of faces one has to traverse in order to close a non-contractible curve (which is a witness for non-planarity of the embedding). The face-width of a graph G is the maximum face-width over all possible embeddings of G. We remark that there is a related parameter, called edge-width, which is the minimum length of a non-contractible cycle.

According to Whitney's theorem, planar 3-connected graphs have a unique embedding in the sphere, but this is not true for arbitrary surfaces. The key result is that a 3-connected graph with large enough face-width has a unique embedding [21]. It turns out that almost all 3-connected graphs have large face-width and, as a consequence, the asymptotic enumeration of 3-connected *graphs* in a surface can be reduced to the enumeration of 3-connected *maps*. In order to enumerate 3-connected maps of genus g we start from the known enumeration of maps of genus g [2,3] via associated quadrangulations.

The final step is to go from 3-connected graphs in a surface to 2-connected, connected and finally arbitrary graphs. Again, the face-width plays the main role in this reduction. A result of Robertson and Vitray [22] says that if a connected graph G of genus g has face-width at least two, then G has a unique block of genus g and the remaining blocks are planar. A similar result holds for 2-connected graphs and 3-connected components. As a consequence, the asymptotic enumeration of graphs of genus g can be reduced to the planar case, which was completely solved in [16].

There is a fundamental difference between the planar and non-planar cases. For planar graphs we have at our disposal *exact* counting generating functions, defined through functional and differential equations [16]. The reason

is precisely that 3-connected graphs have a unique embedding, and there is a bijection with 3-connected maps, for which we know the exact generating function. For higher surfaces this is not the case and we have to *approximate* the counting series. If $f(x)$ is the generating function of interest, we find series $f_1(x)$ and $f_2(x)$ that are computable and whose coefficients have the same leading asymptotic estimates, and such that $f_1(x)$ dominates $f(x)$ coefficient-wise from below, and $f_2(x)$ from above. If we can estimate the coefficients of the $f_i(x)$, then we can estimate those of $f(x)$. A key argument in the proofs is the following. If a graph G of genus g has a short non-contractible cycle C, then cutting G along C produces either a graph of genus $g-1$, or two graphs whose genera add up to g. In either case, induction on the genus g shows that there are few such graphs and that the probability of a graph having a short non-contractible cycle tends to zero as the size of the graph grows. This is precisely where approximate counting series come into play.

2 Random graphs of genus g

In this section we analyze several fundamental parameters of graphs of genus g and derive limit distribution laws for them. In all cases the limit laws do not depend on the genus, a phenomenon that has been observed previously for *maps* on surfaces (see, for instance, [15]). When we say that an event holds *with high probability* we mean that the probability of the event tends to 1 as n tends to infinity. The variance of a random variable X is denoted by $\sigma^2(X)$, and a sequence of random variables $(X_n)_{n\geq 1}$ is called *asymptotically normal* if $(X_n - \mathbf{E}[X_n])/\sigma^2(X_n)$ converges in distribution to a standard Gaussian random variable $\mathcal{N}(0,1)$ (see [14, Part C]).

We start with two basic parameters in order to motivate the general analysis. We show that, as for planar graphs, the number of edges is asymptotically normal and the number of connected components is asymptotically Poisson distributed.

Theorem 3. *The number of edges X_n in a random graph of fixed genus g with n vertices is asymptotically normal and*

$$\mathbf{E}(X_n) \sim \kappa n, \qquad \sigma^2(X_n) \sim \lambda n$$

where $\kappa \approx 2.21326$ and $\lambda \approx 0.43034$ are the same constants as for planar graphs.

One can also show that several basic parameters, such as the number of blocks, the number of cut vertices, and the number of *appearances* of a fixed planar graph (see [16] for a precise definition), follow a normal law with the same moments as for planar graphs [18]. We remark in particular that, given a fixed planar graph H, a random graph of genus g contains a subgraph isomorphic to H with high probability.

Theorem 4. *The number of connected components in a random graph of fixed genus g is distributed asymptotically as $1 + X$, where X is a Poisson law with parameter $\nu \approx 0.037439$, same as for planar graphs. In particular, the probability that a random graph of genus g is connected is asymptotically $e^{-\nu}$.*

A similar analysis as in the proofs of the previous theorem shows that there is a unique giant component of genus g.

Theorem 5. *Let L_n denote the size of the largest connected component in a random graph of fixed genus g with n vertices, and let $M_n = L_n - n$ be the number of vertices not in the largest component. Then*

$$\mathbf{P}(M_n = k) \sim p \cdot g_k \frac{\gamma^{-k}}{k!},$$

where p is the probability of a random planar graph being connected, g_k is the number of planar graphs with k vertices, and γ is the planar growth constant.

It must stressed that, as a consequence, the expected size of the largest component is $n - c$, where $c \approx 0.038$; in other words, it contains everything except a few vertices.

In the next two results we analyze the size of the largest block and the size of the largest 3-connected component. For the precise form of the Airy law of map type, a continuous distribution defined in terms of the Airy function, and the computation of the parameters for planar graphs, we refer to [1] and [18, Section 5]. The main consequence is that with high probability there exists a (unique) block of linear size, containing about 96% of the vertices.

Theorem 6. *The size X_n of the largest block in a random connected graph of fixed genus g with n vertices follows asymptotically an Airy law of the map type, with the same parameters as for planar graphs. In particular*

$$\mathbf{E}(X_n) \sim \alpha n,$$

where $\alpha \approx 0.9598$, and the size of the second largest block is $o(n^{\frac{2}{3}+\epsilon})$, for any $\epsilon > 0$. Moreover the largest block has genus g with high probability.

We can also adapt the proof in [18] for the largest 3-connected component, which is technically more demanding that the previous one.

Theorem 7. *The size X_n of the largest 3-connected component in a random connected graph of fixed genus g with n vertices follows asymptotically an Airy law of the map type, with the same parameters as for planar graphs. In particular*

$$\mathbf{E}(X_n) \sim \alpha_2 n,$$

where $\alpha_2 \approx 0.7346$, and the size of the second largest 3-connected component is $o(n^{\frac{2}{3}+\epsilon})$, for any $\epsilon > 0$. Moreover the largest 3-connected component has genus g with high probability.

We conclude with the chromatic number of a random graph of genus g. We know that asymptotically almost surely a random graph of genus g has face-width greater than any fixed number $k(g)$. Taking $k(g) = 2^{14g+6}$ this implies that it is 5-colorable by a result of Thomassen [23]. Moreover, as mentioned just before Theorem 4, a random graph of genus g has a linear number of copies of K_4 with high probability, in particular has chromatic number at least 4. Consequently:

Theorem 8. *The chromatic number of a random graph of fixed genus g with n vertices is asymptotically almost surely in $\{4, 5\}$.*

Unfortunately, we do not know if both values 4 and 5 appear on a positive proportion of graphs of genus g. However, we conjecture the following:

Conjecture 1. The chromatic number of a random graph of fixed genus g with n vertices is asymptotically almost surely equal to 4.

More precise results hold for the list-chromatic number: as shown in [9], a graph of fixed genus is 5-choosable provided its face-width is large enough. Moreover, there exist planar graphs that are not 4-choosable [25]. If we fix any of them, then it is asymptotically almost surely contained in a random graph of genus g. Therefore we have:

Theorem 9. *The list-chromatic number of a random graph of fixed genus g with n vertices is asymptotically almost surely equal to 5.*

All these result also hold for the random graph of fixed non-orientable genus h, with exactly the same ingredients. Since a random graph with n vertices embeddable on $\mathbb{S}_g$ is asymptotically almost surely of genus g, and a random graph with n vertices embeddable on $\mathbb{N}_h$ is asymptotically almost surely of non-orientable genus h, these results hold also for the random graph embeddable on $\mathbb{S}_g$ and for the random graph embeddable on $\mathbb{N}_h$.

3 Related results

For planar graphs (the $g = 0$ case) there are additional results for extremal parameters of interest, like the maximum degree and the diameter. It is likely that all of them extend to graphs of fixed genus, with the same constants.

We start with the degree distribution in random planar graphs, where many results have been obtained by Drmota, Giménez and Noy.

Theorem 10 ([11]). *For every $k \geq 1$, the probability that a random vertex in a random planar graph has degree k tends to a constant $d_k > 0$, as $n \to \infty$. The tail of the limiting distribution satisfies*

$$d_k \sim c \cdot k^{-1/2} q^k, \qquad k \to \infty,$$

where c and $q \approx 0.67345$ are computable constants.

The probability generating function $p(w) = \sum_{k\geq 1} d_k w^k$ is computable, although the explicit expression is extremely long to write down (see the Appendix in [11]).

The previous result implies that the expected number of vertices of degree k is asymptotically $d_k n$. For series-parallel graphs (a subclass of planar graphs) it has been shown [10] that the number of vertices of vertices of degree k is asymptotically normal with linear expectation and variance, but for planar graphs this remains an open problem.

McDiarmid and Reed [20] have shown that the maximum degree Δ_n of a random planar graph satisfies

$$c \log n < \Delta_n < C \log n$$

with high probability, for suitable constants $0 < c < C$. The precise constant has been determined recently.

Theorem 11 ([13]). *With high probability the maximum degree Δ_n of a random planar graph with n vertices satisifes*

$$\frac{\Delta_n}{\log n} \to \frac{1}{\log 1/q} \approx 2.52946,$$

where q is as in the previous theorem.

An analogous result holds for series-parallel graphs [12].

Finally, very recently we have obtained the order of magnitude of the diameter of random planar graphs, which turns out to be roughly $n^{1/4}$.

Theorem 12 ([7]). *With high probability the Diameter D_n of a connected random planar graph with n vertices satisfies*

$$n^{1/4-\epsilon} \leq D_n \leq n^{1/4+\epsilon},$$

for every ϵ small enough and $n \geq n_0(\epsilon)$.

The proof is based on an analogous (but much more precise) result for planar quadrangulations, due to Chassaing and Schaeffer [8]. From this one can prove the same result for 2-connected maps using the existence of a giant 2-connected component and the fact that the maximum degree is of logarithmic order. The same applies to 3-connected maps, and then we work our way down to 2-connected and connected planar graphs.

As mentioned above, we believe that all these results extend to graphs of arbitrary fixed genus. However, this task may take some time, since the proofs of the results for planar graphs are already quite technical.

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COMUNICACIONES Y POSTERS

Tamaño máximo de los grafos sin triángulos ni cuadriláteros ⋆

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Resumen. Para enteros $n \geq 3$ y $\nu \geq n+1$ la máxima cantidad de aristas de un grafo con ν vértices y cintura al menos $n+1$ se denota por $ex(\nu; \{C_3, C_4, \ldots, C_n\})$. Para el caso $n = 3$, este valor es conocido y viene dado por $ex(\nu; \{C_3\}) = \lfloor \nu^2/4 \rfloor$. Sin embargo, para $n \geq 4$, se conocen muy pocos valores de la función extremal. Concretamente, Garnick, Kwong y Lazebnik [J. Graph Theory 17 (1993), pp 633–645] y Garnick y Nieuwejaar [J. Combin. Math. Combin. Comput. 12 (1992), pp 33–56] determinan el valor de $ex(\nu; \{C_3, C_4\})$ para todo $\nu \leq 30$ y $\nu = 50$. Para esta misma función se conocen cotas inferiores para órdenes $31 \leq \nu \leq 200$, obtenidas a partir de distintos algoritmos que construyen grafos densos libres de triángulos y cuadriláteros. En este trabajo probamos que $ex(31; \{C_3, C_4\}) = 80$ y $ex(32; \{C_3, C_4\}) = 85$. También mejoramos cotas inferiores conocidas de $ex(\nu; \{C_3, C_4\})$ y proporcionamos nuevas cotas de esta función para órdenes $201 \leq \nu \leq 253$.

Palabras clave: Función extremal, grafo extremal, cintura, jaula.

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L-formas asociadas a 3-semigrupos numéricos

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Resumen. Es un resultado conocido que todo semigrupo numérico generado por tres elementos tiene asociadas teselaciones periódicas del plano. Estas teselaciones se forman a partir de una baldosa en forma de 'L', conocida como L-forma, y que se construye a partir de un digrafo de doble lazo de la siguiente manera:

Sean $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^+$. Un *digrafo pesado de doble lazo*, $G = G(N; a, b; \mathfrak{a}, \mathfrak{b})$, es un grafo dirigido con conjunto de vértices $V(G) = \mathbb{Z}_N$ y conjunto de arcos

$$A(G) = \{[v]_N \xrightarrow{\mathfrak{a}} [v+a]_N, [v]_N \xrightarrow{\mathfrak{b}} [v+b]_N \mid [v]_N \in V(G)\},$$

donde $[v]_N$ es la clase de equivalencia de v ódulo N y donde $\xrightarrow{\mathfrak{a}}$ y $\xrightarrow{\mathfrak{b}}$ significa que los arcos correspondientes tienen pesos $\mathfrak{a}$ y $\mathfrak{b}$, respectivamente. Llamaremos *longitud* ℓ de un camino dirigido

$$[v_1]_N \xrightarrow{p_1} [v_2]_N \xrightarrow{p_2} \cdots \xrightarrow{p_{k-1}} [v_k]_N$$

al valor $\ell = p_1 + \cdots + p_{k-1}$.

En el problema de hallar *caminos mínimos* (de longitud mínima) entre dos vértices de G, aparecen las llamadas *teselaciones del plano* mediante *L-formas* ([4,2]). Cada cuadrado unitario del plano con coordenadas (i, j) se asocia al vértice $[ia + jb]_N$ del digrafo G alcanzado por un camino de longitud $i\mathfrak{a} + j\mathfrak{b}$ desde el vértice $[0]_N$.

Si todos los caminos indicados por la L-forma son mínimos, decimos que esta L-forma es un *diagrama de distancias mínimas* (DDM). Sólo se conoce una caracterización geométrica de los DDM planos dada por Rødseth en 1978 en [3], quien probó que los DDM tienen forma de L o son rectangulares.

La notación estándar de una L-forma, $L(l, h, w, y)$, incluye las longitudes de los cuatro lados que la definen. Podemos suponer, sin pérdida de generalidad, que $0 \leq w < l$ y $0 \leq y < h$. Se ha demostrado ([4,2]) que cada L-forma asociada a un digrafo $G(N; a, b; \mathfrak{a}, \mathfrak{b})$ tesela periódicamente el plano mediante las *condiciones de compatibilidad*

$$lh - wy = N, \quad \text{mcd}(l, h, w, y) = 1, \quad la - yb \equiv 0 \pmod{N}, \quad -wa + hb \equiv 0 \pmod{N}.$$

Por otro lado, a cada 3-semigrupo numérico $S = \langle a, b, N\rangle = \{\alpha a + \beta b + \gamma N \mid \alpha, \beta, \gamma \in \mathbb{N}\}$. se le asocian teselaciones del plano por medio de L-formas que son DDM del digrafo digrafo de doble lazo $G(N; a, b; a, b)$.

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En este trabajo se dan algunos resultados relacionados con las L-formas y se utilizan para describir algunas propiedades combinatorias del semigrupo numérico por medio de propiedades geométricas de una L-forma asociada. En particular, se prueba que cada 3-semigrupo numérico tiene asociadas, a lo sumo, dos teselaciones del plano y, cuando son dos, se dan transformaciones geométricas que pasan de una teselación a la otra.

Nuestros principales resultados son:

Teorema 1 ([1]). *Sea $\mathcal{H} = L(l,h,w,y)$ una L-forma asociada a $G = G(N;a,b;a,b)$. Entonces $\mathcal{H}$ es un DDM para G si, y sólo si, $la \geq yb$ y $hb \geq wa$ y además las igualdades no se dan simultáneamente.*

Teorema 2. *Sea $\mathcal{H} = L(l,h,w,y)$ un DDM asociado a $G = G(N;a,b;a,b)$ con $(la-yb)(hb-wa) > 0$. Entonces no existe otro DDM asociado a G.*

Proposición 1. *Sea $\mathcal{H} = L(l,h,w,y)$ un DDM associado a $G = G(N;a,b;a,b)$. Si $hb = wa$, entonces G también tiene asociado el DDM $\mathcal{H}'$ definido por*

$$\mathcal{H}' = \begin{cases} L(w, 2h-y, 2w-l, h) & l < 2w, \\ L(w, (\lfloor l/w \rfloor + 1)h - y, w - r, h) & l > 2w > 0, l = \lfloor l/w \rfloor w + r, 0 < r < w, \\ L(w, lh/w - y, 0, h) & l \geq 2w > 0, l = \lfloor l/w \rfloor w. \end{cases}$$

y análogamente cuando $la = yb$

$$\mathcal{H}' = \begin{cases} L(2l-w, y, l, 2y-h) & h < 2y, \\ L((\lfloor h/y \rfloor + 1)l - w, y, l, y - r) & h > 2y > 0, h = \lfloor h/y \rfloor y + r, 0 < r < y, \\ L(lh/y - w, y, l, 0) & h \geq 2y > 0, h = \lfloor h/y \rfloor y. \end{cases}$$

Corolario 1. *Sea $\mathcal{H} = L(l,h,w,y)$ una L-forma asociada a $S = \langle a,b,N \rangle$ con $(la-yb)(hb-wa) > 0$. Entonces $\mathcal{H}$ es la única L-forma asociada a S.*

Teorema 3. *Sea $\mathcal{H} = L(l,h,w,y)$ una L-forma asociada a S con $(la-yb)(hb-wa) = 0$. Entonces S tiene exactamente dos L-formas asociadas.*

Finalmente, hemos utilizado los resultados anteriores para dar una caracterización de los 3-semigrupos numéricos simétricos a partir de sus L-formas asociadas.

Teorema 4. *Sea $S = \langle a,b,N \rangle$ un semigrupo numérico y $\mathcal{H} = L(l,h,w,y)$ una L-forma asociada a S. Entonces S es simétrico si, y sólo si, $wy = 0$ o $(la-yb)(wa-hb) = 0$.*

Palabras clave: Digrafo de doble lazo, diagrama de distancias mínimas, baldosa en forma de L, L-forma, semigrupo numérico.

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Regular parcial linear spaces admitting $(1; \leq k)$-identifying codes

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Abstract. In this paper we give a characterization of k-regular partial linear spaces admitting a $(1, \leq k)$-identifying code. Equivalently, we give a characterization of k-regular bipartite graphs of girth at least six admitting a $(1, \leq k)$-identifying code. Moreover, we present a family of k-regular partial linear spaces on $2(k-1)^2+k$ points and $2(k-1)^2+k$ lines whose incidence graphs do not admit a $(1, \leq k)$-identifying code. Finally, we show that the smallest $(k; 6)$-graphs known up to now for $k-1$ not a prime power admit a $(1, \leq k)$-identifying code.

Key words: identifying codes, partial linear space, bipartite graph, girth

1 Introduction

Let C be a nonempty subset of V. For $X \subseteq V$ the set of vertices $I(C) = I(C; X)$ is defined as follows

$$I(C) = \bigcup_{x \in X} N[x] \cap C.$$

If all the sets $I(C)$ are different for all subset $X \subseteq V$ where $|X| \leq k$, then C is said to be a $(1, \leq k)$-*identifying code* in G. In 1998, Karpovsky, Chakrabarty and Levitin introduced $(1, \leq k)$-identifying codes in graphs. We say that a graph G *admits a* $(1, \leq k)$-*identifying code* if there exists such a code $C \subseteq V$ in G. Not all graphs admit $(1, \leq k)$-identifying codes, for instance Laihonen pointed out that a graph formed by a set of independent edges cannot admit a $(1, \leq 1)$-identifying code. Laihonen and Ranto proved that if G is a connected graph with at least three vertices admitting a $(1, \leq k)$-identifying code, then the minimum degree is $\delta(G) \geq k$. Gravier and Moncel showed the existence of a graph with minimum degree exactly k admitting a $(1, \leq k)$-identifying code. Recently, Laihonen proved the following result.

Theorem 1. *For an integer $k \geq 2$, if a k-regular graph has girth $g \geq 7$, then it admits a $(1, \leq k)$-identifying code. Moreover, if a k-regular graph has girth $g \geq 5$, then it admits a $(1, \leq k-1)$-identifying code.*

According to Theorem 1, all $(k;6)$-graphs admit a $(1, \leq k-1)$-identifying code. The main aim of this paper is to approach the problem of characterizing bipartite $(k;g)$-graphs for $g \geq 6$ admitting $(1, \leq k)$-identifying codes. To do that we consider a bipartite graph as the incidence graph of a partial linear space $(\mathcal{P}, \mathcal{L}, I)$. A *partial linear space* is an incidence structure in which any two points of $\mathcal{P}$ are incident with at most one line of $\mathcal{L}$. This implies that any two lines are incident with at most one point. The incidence graph of a k-regular partial linear space is a k-regular bipartite graph of girth at least 6.

2 Main theorem

Let $(\mathcal{P}, \mathcal{L}, I)$ be a partial linear space and $X \subseteq \mathcal{P} \cup \mathcal{L}$. Following Dembowski, let us denote by $(X)_I = \bigcup_{x \in X} \{y : yIx\}$ and by $[X] = (X)_I \cup X$. With this terminology we give the following definition.

Definition 1. *A partial linear space $(\mathcal{P}, \mathcal{L}, I)$ is said to admit a $(1, \leq k)$-identifying code if and only if the sets $[X]$ are mutually different for all $X \subseteq \mathcal{P} \cup \mathcal{L}$ with $|X| \leq k$.*

Next, we characterize the k-regular partial linear spaces admitting a $(1, \leq k)$-identifying code.

Theorem 2. *Let $k \geq 2$ be an integer. A k-regular partial linear space $(\mathcal{P}, \mathcal{L}, I)$ admits a $(1, \leq k)$-identifying code if and only if the following two conditions hold:*

(i) For every collinear points u, p there exists a point $z \in \mathcal{P}$ which is collinear with just one of u, p.
(ii)For every two concurrent lines L, M there exists a line Λ which is concurrent with just one of L, M.

As a consequence we characterize k-regular bipartite graphs of girth at least 6 admitting $(1, \leq k)$-identifying codes.

Theorem 3. *A $(k;g)$-regular bipartite graph, $g \geq 6$, admits a $(1, \leq k)$-identifying code if and only if for every two vertices u, v with $|N(u) \cap N(v)| = 1$, there exists $z \in V$ such that $|N(u) \cap N(z)| + |N(v) \cap N(z)| = 1$.*

In the rest of the paper we describe families of small $(k, 6)$-graphs admitting a $(1, \leq k)$-identifying code, and families without such a code.

La Función de Green y el Índice de Kirchhoff de Redes Cluster

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Resumen. En este trabajo determinamos las Funciones de Green, respecto de un peso fijado sobre los vértices, en una amplia clase de redes compuestas que se engloban bajo la denominación de redes clúster. También aplicamos las expresiones obtenidas al cálculo de la *resistencia efectiva, respecto de un peso, entre cualquier par de vértices* y el Índice de Kirchhoff de la red *clúster*, en términos de los parámetros correspondientes a cada uno de sus factores. Finalmente mostramos que la particularización de nuestros resultados al caso estándar, esto es, al caso de grafos con peso constante en los vértices recupera las expresiones obtenidas por otros autores en trabajos previos.

El Índice de Kirchhoff fue introducido en Química como una alternativa mejor a otros parámetros usados para discriminar entre diferentes moléculas con formas y estructuras similares. Al calcular este parámetro sobre redes compuestas, se encuentran relaciones entre los Índices de Kirchhoff de la red compuesta y los de sus factores. Algunos autores de este trabajo obtuvieron una generalización del Índice de Kirchhoff, las resistencias totales y las resistencias efectivas respecto de un peso ω de una red finita $\Gamma = (V, c)$ y demostraron que vienen dados por las siguientes igualdades:

$$\mathsf{k}(\omega) = \sum_{x \in V} G_q(x, x) \quad \text{Índice de Kirchhoff,}$$

$$R_\omega(x, y) = \frac{G_q(x, x)}{\omega^2(x)} + \frac{G_q(y, y)}{\omega^2(y)} - \frac{2G_q(x, y)}{\omega(x)\omega(y)} \quad \text{resistencia efectiva,}$$

donde G_q es la función de Green de la red. En este trabajo, esta generalización es esencial para poder obtener la expresión del Índice de Kirchhoff de una red clúster en términos de los Índices de Kirchhoff de los factores. Todas las definiciones (red, conductancias, etc.), notaciones y resultados importantes que aplicaremos se pueden encontrar en la bibliografía del presente trabajo.

En este trabajo consideraremos la siguiente generalización del clúster clásico usado en la literatura: sea $\Gamma_0 = (V_0, E_0, c_0)$, donde $V_0 = \{x_1, \ldots, x_m\}$, una red conexa y para cada $i = 1, \ldots, m$ consideremos $\Gamma_i = (V_i, E_i, c_i)$

una red conexa tal que $x_i \in V_i$. Llamamos *red clúster con base* Γ_0, a la red $\Gamma = (V, c)$ con vértices $V = \coprod_{i=1}^{m} V_i$ y conductancia $c(x,y) = c_i(x,y)$, $\forall x, y \in V_i$, $i = 0, \ldots, m$, $\quad c(x,y) = 0$ de otra forma. Consideramos ω un peso en V y para $i = 0, \ldots, m$ alteramos con coeficientes σ_i el peso ω para obtener pesos derivados de éste, ω_i, en cada factor de la red clúster.

La función de Green de la red clúster $\Gamma_0\{\Gamma_1, \ldots, \Gamma_m\}$ con peso ω está determinada por las identidades

$$\begin{aligned} G_{q_\omega} &= G^j_{q_{\omega_j}(\cdot,x_j)} + \tfrac{\sigma_j^2-1}{\omega_j(x_j)} G^j_{q_{\omega_j}(x_j,\cdot)} \otimes \omega_j + \tfrac{\sigma_j^2-1}{\omega_j(x_j)} \omega_j \otimes G^j_{q_{\omega_j}} + g_j \omega_j \otimes \omega_j, \\ &\qquad\qquad \text{en } V_j \times V_j,\ j = 1, \ldots, m, \\ G_{q_\omega} &= \tfrac{\sigma_i\sigma_j}{\omega_i(x_i)} G^i_{q_{\omega_i}(\cdot,x_i)} \otimes \omega_j + \tfrac{\sigma_i\sigma_j}{\omega_j(x_j)} \omega_i \otimes G^j_{q_{\omega_j}(x_j,\cdot)} + g_{ij} \omega_i \otimes \omega_j, \\ &\qquad\qquad \text{en } V_i \times V_j,\ j = 1, \ldots, m,\ i \neq j. \end{aligned}$$

donde para cada $i, j = 1, \ldots, m$

$$\begin{aligned} g_j = {} & \tfrac{G^0_{q_{\omega_0}}(x_j,x_j)}{\omega_j^2(x_j)} + \tfrac{1-2\sigma_j^2}{\omega_j^2(x_j)} G^j_{q_{\omega_j}}(x_j,x_j) + \sigma_j^2 \sum_{i=1}^{m} \tfrac{\sigma_i^2}{\omega_i(x_i)} G^i_{q_{\omega_i}}(x_i,x_i) - \\ & - \tfrac{2\sigma_j}{\omega_j(x_j)} \sum_{i=1}^{m} \tfrac{\sigma_i}{\omega_i(x_i)} G^0_{q_{\omega_0}}(x_i,x_j) + \sigma_j^2 \sum_{i=1}^{m} \tfrac{\sigma_i}{\omega_i(x_i)} \sum_{k=1}^{m} \tfrac{\sigma_k}{\omega_k(x_k)} G^0_{q_{\omega_0}}(x_i,x_k), \\ g_{ij} = {} & \tfrac{G^0_{q_{\omega_0}}(x_i,x_j)}{\omega_i(x_i)\omega_j(x_j)} - \tfrac{\sigma_i\sigma_j}{\omega_j^2(x_j)} G^j_{q_{\omega_j}}(x_j,x_j) - \tfrac{\sigma_i\sigma_j}{\omega_i^2(x_i)} G^i_{q_{\omega_i}}(x_i,x_i) - \\ & - \tfrac{\sigma_i}{\omega_j(x_j)} \sum_{k=1}^{m} \tfrac{\sigma_k}{\omega_k(x_k)} G^0_{q_{\omega_0}}(x_k,x_j) - \tfrac{\sigma_j}{\omega_i(x_i)} \sum_{k=1}^{m} \tfrac{\sigma_k}{\omega_k(x_k)} G^0_{q_{\omega_0}}(x_k,x_i) + \\ & + \sigma_i\sigma_j \sum_{k=1}^{m} \tfrac{\sigma_k^2}{\omega_k^2(x_k)} G^k_{q_{\omega_k}}(x_k,x_k) + \sigma_i\sigma_j \sum_{k=1}^{m} \tfrac{\sigma_k}{\omega_k(x_k)} \sum_{l=1}^{m} \tfrac{\sigma_l}{\omega_l(x_l)} G^0_{q_{\omega_0}}(x_k,x_l) \end{aligned}$$

Se obtiene a partir de esto la expresión del Índice de Kirchhoff $\mathsf{k}(\omega)$ y de las resistencias efectivas R_ω de una red clúster en función de los mismos parámetros $\mathsf{k}_i(\omega_i)$ y R_{ω_i} de los factores:

$$\mathsf{k}(\omega) = \sum_{i=1}^{m} \mathsf{k}_i(\omega_i) + \frac{1}{2\sigma_0^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_i^2\sigma_j^2 R_{\omega_0}(x_i,x_j) + \sum_{i=1}^{m} \left((1-\sigma_i^2) \sum_{x \in V_i} r_{\omega_i}(x_i) \right),$$

$$\begin{aligned} R_\omega(x,y) &= \tfrac{R_{\omega_i}(x,y)}{\sigma_i^2} \qquad \text{si } x, y \in V_i; \\ R_\omega(x,y) &= \tfrac{R_{\omega_j}(x,x_j)}{\sigma_j^2} + \tfrac{R_{\omega_i}(y,x_i)}{\sigma_i^2} + \tfrac{R_{\omega_0}(x_i,x_j)}{\sigma_0^2} + \tfrac{2\sigma_j^2-1}{\sigma_j^2} r_{\omega_j}(x_j) + \\ &\quad + \tfrac{2\sigma_i^2-1}{\sigma_i^2} r_{\omega_i}(x_i) + 2t_{ij} \qquad \text{si } x \in V_j,\ \ y \in V_i,\ \ i = 1, \ldots, m,\ \ i \neq j. \end{aligned}$$

donde para cada $i, j = 1, \ldots, m$

$$\begin{aligned} t_{ij} = {} & - \sum_{k=1}^{m} \sigma_k^2 r_{\omega_k(x_k)} - \tfrac{1}{\sigma_0^2} \sum_{k=1}^{m} \sigma_k^2 r_{\omega_0(x_k)} - \tfrac{1}{2\sigma_0^2} \sum_{k=1}^{m} \sigma_k^2 R_{\omega_0}(x_k,x_i) - \\ & - \tfrac{1}{2\sigma_0^2} \sum_{k=1}^{m} \sigma_k^2 R_{\omega_0}(x_k,x_j) + \tfrac{1}{2\sigma_0^2} \sum_{k=1}^{m} \sum_{l=1}^{m} \sigma_k^2\sigma_l^2 R_{\omega_0}(x_k,x_l). \end{aligned}$$

Practical challenges that arise when clustering the web using spectral methods

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Extended Abstract

Spectral clustering is a technique for partitioning data based on the spectrum of a similarity matrix: a matrix which registers some pairwise similarity between the data points, and to which one computes its *eigenvalues* and *eigenvectors* and uses these for clustering, or grouping, data as points in a smaller dimension. This data mining method seem to be gaining popularity over other more traditional clustering algorithms, for its robustness and adequacy for effectively classifying large and sparse data sets.

The interest of the authors in understanding the topology of the Web [1,2], and the linear algebra behind spectral graph theory, led them to conduct a student's math and computation project for retrieving and clustering large parts of internet sites, with the main objective of understanding the methods rather than drawing conclusions from the particular classification.

Our main guide for learning about spectral clustering is the fine survey by Luxburg [5], and the fundamental papers of Shi and Malik [4], and Ng, Jordan and Weiss [3]. When we set about implementing the algorithms as explained in these references we ran into unforeseen troubles related to the convergence of the numerical methods used to compute eigenvalues for very large, and possibly sparse, matrices. One of these difficulties arises from the fact that the spectral problem, as stated in the consulted references on spectral clustering, is badly *conditioned*. We will explain what this means, and how to overcome this issue, which is surprisingly overlooked in the literature, although it is inherent to the computational methods involved in spectral clustering. Other difficulties are related with the limitations of the computational system that we used for implementing our algorithmic solution; in our case it is *MatLab* wired-up with C++. We will point out what these are and how to solve them. Also, faced with the challenge of exploring large parts of the internet in an efficient manner, we needed to get hold of a robust and powerful (and preferably, freely available) web crawler. Our selected system was WIRE [6], a professional open-source internet crawler designed for exploring and making statistics of usage of big commu-

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nities web sites (e.g. WIRE has been used for analyzing the full internet domain of Chile and Spain). As magnificent as WIRE can be, it needed some adaptation for our particular purposes, as for example, the addition of the extra functionality of generating the adjacency matrix of the explored domain in the format adequate for our computations. We will indicate the necessary changes and outline our practical steps for implementing a spectral clustering method, and we show the results of our experiments.

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On the restricted edge-connectivity of 3-arc graphs

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Abstract. A graph G is called connected if every pair of vertices is joined by a path. An *edge-cut* in a graph G is a set W of edges of G such that $G-W$ is not connected. The *edge-connectivity* $\lambda(G)$, of a graph G, is the minimum cardinality of an edge cut of G. A graph G is called r-edge-connected if $\lambda(G) \geq r$. It is well known that $\lambda(G) \leq \delta(G)$ where $\delta(G)$ is the minimum degree over all vertices of G. Thus, a graph G is said to be *maximally edge-connected* when $\lambda(G) = \delta(G)$.

Though the parameter λ of edge-connectivity gives the minimum cost to disrupt the network, it does not take into account what remains after deletion. Even two graphs with the same edge-connectivity λ may be considered to have different reliabilities, since the number of minimum edge-cuts is different.

From here arises the notion of *restricted edge-connectivity*. The restricted edge-connectivity $\lambda' = \lambda'(G)$ is the minimum cardinality over all *restricted edge-cuts* W, i.e., those such that there are no isolated vertices in $G-W$. A restricted edge-cut W is called a λ'*-cut* if $|W| = \lambda'$. A connected graph G is called λ'*-connected* if λ' exists. Esfahanian and Hakimi showed that each connected graph G of order $n(G) \geq 4$ except a star, is λ'-connected and satisfies $\lambda' \leq \xi$, where $\xi = \xi(G)$ denotes the *minimum edge-degree* of G defined as $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$. Furthermore, a λ'-connected graph is said to be λ'*-optimal* if $\lambda' = \xi$.

The purpose of this work is to study the edge-connectivity and restricted edge-connectivity of the 3-arc graphs of a given graph G. We proved that this kind of graphs have high edge-connectivity and restricted edge-connectivity. The 3-arc graph was introduced by Zanming Zhou in 2002.

Let $\overleftrightarrow{G}$ denote the symmetric digraph of a graph G. For adjacent vertices u, v of G we use (uv) to denote the arc from u to v, $(vu)(\neq (uv))$ to denote the arc from v to u, and $uv = vu$ the edge between u and v. A *3-arc* is a 4-tuple (y, a, b, x) of vertices such that both (y, a, b) and (a, b, x) are paths of length two in G. The *3-arc graph* $X(G)$ of a given graph G is defined to have vertices the arcs of $\overleftrightarrow{G}$. Two vertices $(ay), (bx)$ are adjacent in $X(G)$ if and only if (y, a, b, x) is a 3-arc of G. Equivalently, two vertices $(ax), (by)$ are adjacent in $X(G)$ if and only if $d_G(a, b) = 1$; that is, the tails a, b of the arcs $(a, x), (b, y) \in A(\overleftrightarrow{G})$ are at distance one in G. Thus the number of edges of $X(G)$ is $\sum_{uv \in E(G)} (d(u) - 1)(d(v) - 1)$ so that the minimum degree of $X(G)$ is $(\delta(G) - 1)^2$. There is a bijection between the edges of $X(G)$ and those of the 2-path graph $P_2(G)$, which is defined to have vertices the paths of length two in

G such that two vertices are adjacent if and only if the union of the corresponding paths is a path or a cycle of length three. Since $P_2(G)$ is a spanning subgraph of the second iterated line graph $L_2(G) = L(L(G))$, we have a relation between 3-arc graphs and line graphs.

In 2010, M. Knor and S. Zhou studied its vertex connetivity. The *vertex connectivity* $\kappa = \kappa(G)$ of a graph G, is the minimum cardinality of an vertex cut of G and G is called r-connected if $\kappa \geq r$. They proved that the 3-arc graph $X(G)$ is connected if G is a connected graph with minimum degree $\delta(G) \geq 3$ and $X(G)$ is $(\kappa-1)^2$-connected if G is κ-connected with $\kappa \geq 3$.

In this work, we prove that the 3-arc graph $X(G)$ of every λ-connected graph G has high connectivity. Given a connected graph G of minimum degree $\delta(G) \geq 3$, we prove that $X(G)$ is $(\delta(G) - 1)^2$-connected. Furthermore if G is a graph with minimum degree $\delta(G) \geq 3$ and vertex connectivity $\kappa \geq 2$, then $X(G)$ has restricted edge-connectivity $\lambda'(X(G)) \geq 2(\delta(G) - 1)^2 - 2$. As corollaries we obtain that $X(G)$ is maximally edge-connected if G is a connected and regular graph with minimum degree $\delta(G) \geq 3$ and even more, if G has vertex connectivity $\kappa \geq 2$, then $X(G)$ is λ'-optimal, i.e., $\lambda'(X(G)) = \xi(X(G)) = 2(r - 1)^2 - 2$. We also provide examples showing that all these bounds are sharp.

Arte y algoritmos ⋆

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Pocos años después de la aparición, en la década de 1940-50, de los primeros ordenadores unos pocos artistas empezaron a interesarse por el uso del ordenador como herramienta para la creación y a explorar sus capacidades expresivas. Con las primeras obras creadas con ordenador, entre las que cabe destacar *Oscillons*, de Ben Laposki, en 1952, la película *Catalog*, de John Withney, en1962, las obras tempranas de Manfred Mohr, aun en activo, que empezó a usar algoritmos para sus creaciones gráficas en 1969, se puso de manifiesto una nueva forma de creación que fue llamada *arte digital* o *computer art*. Algunos de los primeros artistas digitales fueron también ingenieros o científicos, como los artistas pertenecientes al grupo de Stuttgart Frieder Nake, estudiante de matemáticas que obtuvo permiso de la universidad para experimentar con un Zuse Z22, uno de los primeros ordenadores, durante la noche, o Georg Ness, que había estudiado matemáticas y física, además de filosofía. En España se creó un grupo de artistas digitales alrededor del seminario de Generación Automática de Formas Plásticas del Centro de Cálculo de la Universidad de Madrid.

Para una introducción a las relaciones entre informática y arte ver [1]. Las diferencias entre arte generativo y *software art* se discuten en [2].

1 Arte generativo

Según Galanter [4], el arte generativo es cualquier práctica artística en la cual el artista utiliza un sistema, como un conjunto de reglas de un lenguaje natural, un programa informático, una máquina o cualquier otro procedimiento inventado, que se activa con un cierto grado de autonomía , contribuyendo a la creación de una obra de arte completa. La definición describe procesos definidos por reglas. Al arrancar estos processos autoorganizados, sujetos a reglas o instrucciones predefinidas, funcionan independientemente de sus autores o artistas programadores. Interesa sobretodo la potencial multiplicidad de resultados y un cierto grado de impredictibilidad. Pero no se trata necesariamente de arte computacional.

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1.1 Sistemas generativos

Los sistemas generativos abarcan un amplio espectro, desde sistemas deterministas con alto grado de simetría a sistemas caóticos o procesos aleatorios. Así, las reglas que definen un proceso generativo pueden basarse en modelos estáticos y deterministas, usando por ejemplo, patrones simétricos, de tipo geométrico; combinatoria; fractales. También pueden surgir de sistemas dinámicos, que incorporen un cierto grado de aleatoriedad, como modelos caóticos o modelos de la termodinámica. O bien de modelos que imiten algunas cualidades de lo vivo o que imiten aspectos de la creación a partir del lenguaje [3]: sistemas de Lindenmayer; autómatas celulares, máquinas de estado finito; redes neuronales y lógica difusa; algoritmos genéticos; sistemas complejos autoorganizados.

2 *Software art*

El *software art*, término acuñado en 2001, se basa en considerar el *software* como material artístico, y no como herramienta. En este contexto, el arte no sólo se realiza mediante el ordenador, sino que tiene lugar en el ordenador. El *software* no es únicamente un instrumento funcional, también se puede considerar una creación en sí misma, donde el material estético resultante es el código generado y la forma de expresión es la programación. Encontramos aquí dos líneas de trabajo: de un lado, el trabajo con el código en sí mismo, del otro, el trabajo crítico entorno de la vertiente cultural de los programas.

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The M–matrix Moore–Penrose inverse problem for weighted paths

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Key words: M–matrix, Moore–Penrose inverse, Laplacian, tridiagonal–matrix.

1 Extended Abstract

The matrices that can be expressed as $L = kI - A$, where $k > 0$ and $A \geq 0$, appear in relation with systems of equations or eigenvalue problems in a broad variety of areas including finite difference methods for solving partial differential equations, input–output production and growth models in economics or Markov processes in probability and statistics. Of course, the combinatorial community can recognize within this type of matrices, the combinatorial Laplacian of a k–regular graph where A is the adjacency matrix.

If k is at least the spectral radious of A, then L is called a *M–matrix*.

A well–known property of an irreducible non–singular M–matrix is that its inverse is non–negative. However, when the matrix is an irreducible and singular M–matrix it is known that it has a generalized inverse which is non–negative, but this is not always true for any generalized inverse. For instance, it may happens that the Moore–Penrose inverse has some negative entries. We focus here in characterizing when the Moore–Penrose inverse of a symmetric, singular and irreducible and tridiagonal M–matrix is itself a M–matrix. This problem has been widely studied for several types of matrices.

Given $c_1, \ldots, c_{n-1} > 0$ and $d_1, \ldots, d_n \geq 0$, the tridiagonal matrix

$$\mathsf{M} = \begin{bmatrix} d_1 & -c_1 & & & \\ -c_1 & d_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & d_{n-1} & -c_{n-1} \\ & & & -c_{n-1} & d_n \end{bmatrix} \tag{1}$$

is a singular M–matrix iff there exists $\omega_1, \ldots, \omega_n > 0$ with $\omega_1^2 + \ldots + \omega_n^2 = 1$ and

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$$d_1 = \frac{c_1\omega_2}{\omega_1}, \quad d_n = \frac{c_{n-1}\omega_{n-1}}{\omega_n} \quad \text{and} \quad d_j = \frac{1}{\omega_j}(c_j\omega_{j+1} + c_{j-1}\omega_{j-1}) \tag{2}$$

for any $j = 2, \ldots, n-1$. We aim here at determining when $\mathsf{M}^\dagger$, the Moore–Penrose inverse on M, is also a M–matrix.

Proposition 1. *The Moore–Penrose inverse of* M *is* $\mathsf{M}^\dagger = (g_{ij})$*, where*

$$g_{ji} = g_{ij} = \omega_i\omega_j \left[\sum_{k=1}^{i-1} \frac{\left(\sum_{l=1}^{k} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}} + \sum_{k=i}^{n-1} \frac{\left(\sum_{l=k+1}^{n} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}} - \sum_{k=i}^{j-1} \frac{\left(\sum_{l=k+1}^{n} \omega_l^2\right)}{c_k\omega_k\omega_{k+1}} \right]$$

for any $1 \leq i \leq j \leq n$.

If we take into account that the Moore–Penrose inverse of a symmetric and positive semi–definite matrix is itself symmetric and positive semi–definite, as a by–product of the expression of $\mathsf{M}^\dagger$ we can easily characterize when it is a M–matrix.

Theorem 1. $\mathsf{M}^\dagger$ *is a* M*–matrix iff* $g_{ii+1} \leq 0$ *for any* $i = 1, \ldots, n-1$*; that is, iff*

$$\frac{\left(\sum_{l=i+1}^{n} \omega_l^2\right)\left(\sum_{l=1}^{i} \omega_l^2\right)}{c_i\omega_i\omega_{i+1}} \geq \sum_{k=1}^{i-1} \frac{\left(\sum_{l=1}^{k} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}} + \sum_{k=i+1}^{n-1} \frac{\left(\sum_{l=k+1}^{n} \omega_l^2\right)^2}{c_k\omega_k\omega_{k+1}}, \quad i = 1, \ldots, n-1.$$

We characterize here when given $\omega_1, \ldots, \omega_n > 0$ with $\omega_1^2 + \ldots + \omega_n^2 = 1$ there exist $c_1, \ldots, c_{n-1} > 0$ such that if $d_1, \ldots, d_{n-1}$ satisfy Identity (2), then the Moore–Penrose inverse of matrix M in (1) is a M–matrix.

Optimal codes over $\mathbb{Z}_2 \times \mathbb{Z}_4$ *

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Abstract. We study additive codes over $\mathbb{Z}_2 \times \mathbb{Z}_4$ with largest minimum distance. We find two kinds of maximum distance separable codes and we state which are the possible parameters, i.e. the type of the code, for such codes.

Key words: Additive codes, minimum distance bounds, maximum distance separable codes.

1 Introduction

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, which is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Such a code $\mathcal{C}$ is isomorphic to an abelian structure $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. Therefore, $\mathcal{C}$ is of type $2^\gamma 4^\delta$ as a group, it has $|\mathcal{C}| = 2^{\gamma+2\delta}$ codewords and the number of order two codewords in $\mathcal{C}$ is $2^{\gamma+\delta}$. Let X (respectively Y) be the set of $\mathbb{Z}_2$ (respectively $\mathbb{Z}_4$) coordinate positions, so $|X| = \alpha$ and $|Y| = \beta$. Call $\mathcal{C}_X$ (respectively $\mathcal{C}_Y$) the punctured code of $\mathcal{C}$ by deleting the coordinates outside X (respectively Y). Let $\mathcal{C}_b$ be the subcode of $\mathcal{C}$ which contains all order two codewords and let κ be the dimension of $(\mathcal{C}_b)_X$, which is a binary linear code. For the case $\alpha = 0$, we will write $\kappa = 0$. Considering all these parameters, we will say that $\mathcal{C}$ (or equivalently $C = \Phi(\mathcal{C})$) is of type $(\alpha, \beta; \gamma, \delta; \kappa)$.

2 Bounds on the minimum distance

Theorem 1. *If $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with parameters $(\alpha, \beta, \gamma, \delta, \kappa)$, then*

$$\frac{d(\mathcal{C}) - 1}{2} \leqslant \frac{\alpha}{2} + \beta - \frac{\gamma}{2} - \delta, \tag{1}$$

$$\left\lfloor \frac{d(\mathcal{C}) - 1}{2} \right\rfloor \leqslant \alpha + \beta - \gamma - \delta. \tag{2}$$

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Theorem 2. *If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code which is separable, i.e. $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$, then the minimum distance is given by*

$$d(\mathcal{C}) = \min\{d(\mathcal{C}_X), d(\mathcal{C}_Y)\}. \tag{3}$$

Corollary 1. *If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ which is separable, then*

$$d(\mathcal{C}) \leq \min\left\{\alpha - \kappa + 1, \overline{d}\right\}, \tag{4}$$

where $\overline{d}$ is the maximum value satisfying both Bound (1) and Bound (2).

3 Maximum distance separable codes

We say that a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is maximum distance separable (MDS) if $d(\mathcal{C})$ meets the bound given in (1) or (2).

Proposition 1. *If $\mathcal{C}$ is an MDS $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, $\alpha > 0$, such that $d(\mathcal{C})$ meets Bound (2), then $d(\mathcal{C})$ is odd and $\alpha = 1$.*

Proposition 2. *If $\mathcal{C}$ is an MDS code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ satisfying Bound (1), then $\gamma \leqslant 1$.*

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Nuevas cotas superiores de algunos números de Ramsey del tipo $r(K_m, K_n - e)$.

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Resumen. Para los casos no triviales, es decir $n \geq 4$ y $m \geq 3$, son conocidos los valores exactos de los números de Ramsey del tipo $r(K_m, K_n - e)$ únicamente cuando o bien $n = 4$ y $3 \leq m \leq 6$ o bien $n = 5$ y $3 \leq m \leq 4$ o bien $6 \leq n \leq 9$ y $m = 3$ y también se conocen tanto cotas superiores como inferiores para otros valores.

En este trabajo se obtienen nuevas cotas superiores de dichos números cuando $4 \leq m \leq 8$ y $4 \leq n \leq 11$. Entre ellas, merecen una mención especial $r(K_4, K_6 - e) \leq 35$, $r(K_6, K_5 - e) \leq 53$, $r(K_7, K_4 - e) \leq 31$ y $r(K_8, K_4 - e) \leq 42$.

1 Introducción

Dados dos grafos G_1 y G_2 y un número natural k, denotaremos como $R(G_1, G_2; k)$ al conjunto de los grafos de orden k que ni contienen a G_1 ni su complementario contiene a G_2. El número de Ramsey $r(G_1, G_2)$ es el menor natural r tal que $R(G_1, G_2; r) = \emptyset$.

En este trabajo se mejorarán algunas cotas superiores de los números de Ramsey del tipo $r(K_m, K_n - e)$.

Además de los valores obvios $r(K_m, K_3 - e) = 2m - 1$ y $r(K_2, K_m - e) = m$, son conocidos únicamente los valores exactos $r(K_3, K_4 - e) = 7$, $r(K_3, K_5 - e) = 11$, $r(K_3, K_6 - e) = 17$, $r(K_3, K_7 - e) = 21$, $r(K_3, K_8 - e) = 25$, $r(K_3, K_9 - e) = 31$, $r(K_4, K_4 - e) = 11$, $r(K_4, K_5 - e) = 19$, $r(K_5, K_4 - e) = 16$ y $r(K_6, K_4 - e) = 21$.

En cuanto a las cotas superiores de otros valores de m y n son conocidas $r(K_3, K_{10} - e) \leq 38$, $r(K_3, K_{11} - e) \leq 47$, $r(K_4, K_6 - e) \leq 36$[1], $r(K_4, K_7 - e) \leq 52$, $r(K_5, K_5 - e) \leq 34$, $r(K_5, K_6 - e) \leq 68$, $r(K_4, K_7 - e) \leq 112$, $r(K_6, K_5 - e) \leq 55$[1], $r(K_6, K_6 - e) \leq 116$[1], $r(K_6, K_7 - e) \leq 205$, $r(K_7, K_4 - e) \leq 34$[1], $r(K_7, K_5 - e) \leq 88$[1], $r(K_7, K_6 - e) \leq 202$ y $r(K_8, K_4 - e) \leq 45$[1] (ver [3]).

También existen fórmulas generales que proporcionan una cota superior para cualquier valor de m y cualquier valor de n en función de valores más pequeños.

2 Resultados obtenidos

En este trabajo se obtienen nuevas cotas superiores de los números de Ramsey del tipo $r(K_m, K_n - e)$ cuando $4 \leq m \leq 8$ y $4 \leq n \leq 11$, entre las cuales hay que destacar $r(K_4, K_6 - e) \leq 35$, $r(K_6, K_5 - e) \leq 53$, $r(K_7, K_4 - e) \leq 31$ y $r(K_8, K_4 - e) \leq 42$.

[1] Cota mejorada en este trabajo.

Utilizando cotas superiores ya conocidas en la literatura, las cuatro cotas anteriores y fórmulas que acotan superiormente el número de Ramsey de dos grafos en función de números de Ramsey de grafos más pequeños se obtienen nuevas cotas.

Las fórmulas utilizadas son:

1. [1] $r(K_n, K_m - e) \leq r(K_{n-1}, K_m - e) + r(K_n, K_{m-1} - e)$. La desigualdad es estricta si ambos sumandos del miembro derecho son pares.

   Sean $\alpha \geq r(K_n, K_{m-2} - e) - 1$, $\beta \geq r(K_{n-2}, K_m - e) - 1$, $\gamma \geq r(K_n, K_{m-1} - e) - 1$ y $\delta \geq r(K_{n-1}, K_m - e) - 1$. Se tiene que:

2. [2] Si $\gamma(4\alpha + 2\beta - 3\gamma + 6) + (\beta + 1)^2 \geq 0$ entonces $r(K_n, K_m - e) \leq \text{máx}\{2\gamma + 2 + \frac{1}{3}(\beta - \alpha), \frac{1}{2}(\beta + 3\gamma + 5)\sqrt{\gamma(4\alpha + 2\beta - 3\gamma + 6) + (\beta + 1)^2}\}$.

3. [2] Si $\delta(4\beta + 2\alpha - 3\delta + 6) + (\alpha + 1)^2 \geq 0$ entonces $r(K_n, K_m - e) \leq \text{máx}\{2\delta + 2 + \frac{1}{3}(\alpha - \beta), \frac{1}{2}(\alpha + 3\delta + 5)\sqrt{\delta(4\beta + 2\alpha - 3\delta + 6) + (\alpha + 1)^2}\}$.

4. [2] $r(K_n, K_m - e) \leq \alpha + \beta + 4 + 2\sqrt{\alpha + \beta + 1 + \frac{1}{3}(\alpha^2 + \alpha\beta + \beta^2)}$.

Las cotas obtenidas, así como las conocidas con anterioridad a este trabajo se muestran en la siguiente tabla, donde $*$ significa que se mejora la hasta ahora mejor cota conocida y $**$ que es la primera vez que aparece en la literatura una cota superior del número de Ramsey de dicha pareja de grafos:

	$K_3 - e$	$K_4 - e$	$K_5 - e$	$K_6 - e$	$K_7 - e$	$K_8 - e$	$K_9 - e$	$K_{10} - e$	$K_{11} - e$
K_3	5	7	11	17	21	25	31	38	47
K_4	7	11	19	35^*	52	77^{**}	105^{**}	143^{**}	187^{**}
K_5	9	16	34	68	112	186^{**}	277^{**}	418^{**}	586^{**}
K_6	11	21	53^*	114^*	205	385^{**}	621^{**}	1035^{**}	1551^{**}
K_7	13	31^*	84^*	197^*	394^{**}	768^{**}	1339^{**}	2355^{**}	3766^{**}
K_8	15	42^*	123^{**}	306^{**}	659^{**}	1382^{**}	2562^{**}	4844^{**}	8223^{**}

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The Frobenius problem for numerical semigroups

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In this talk, we characterize those numerical semigroups containing $\langle n_1, n_2 \rangle$. From this characterization, we give formulas for the genus and the Frobenius number of a numerical semigroup. These results can be used to give a method for computing the genus and the Frobenius number of a numerical semigroup with embedding dimension three in terms of its minimal system of generators.

Key words: numerical semigroup, Frobenius number, genus.

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Rotational and dihedral symmetries in Steinhaus and Pascal binary triangles ⋆

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Key words: Steinhaus triangles, Pascal triangle, Symmetric arrangements, Rotational symmetry, Dihedral symmetry

Let F_2 be the binary field. The *Steinhaus triangle* of the sequence $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in F_2^n$ is the sequence $(\mathbf{x}, \partial\mathbf{x}, \ldots, \partial^{n-1}\mathbf{x})$, where $\partial\mathbf{x} = (x_0{+}x_1,\ x_1{+}x_2, \ldots,\ x_{n-2}{+}x_{n-1})$ and $\partial^i\mathbf{x} = \partial\partial^{i-1}\mathbf{x}$ for $2 \le i \le n-1$. Figure on the left of figure 1 shows a graphical representation of $S(\mathbf{x})$ for the sequence $\mathbf{x} = (0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0)$. The black and white circles represent ones and zeroes respectively; the first row corresponds to $\mathbf{x}$ and the following rows to the iterated derivatives. Each entry of the triangle is the binary sum of the two values immediately above it.

Let $\mathbf{u} = (u_0,\ \ldots,\ u_\ell)$ and $\mathbf{v} = (v_0,\ \ldots,\ v_\ell)$ be two binary sequences in $F_2^{\ell+1}$ with $u_0 = v_0$. The *Pascal triangle* is the double indexed sequence $\mathbf{z}(r, c)$ defined by the initial conditions $\mathbf{z}(r, 0) = u_r$ and $\mathbf{z}(r, r) = v_r$, for $0 \le r \le \ell$, and the recurrence $\mathbf{z}(r, c) = \mathbf{z}(r-1, c-1) + \mathbf{z}(r-1, c)$, $1 \le r \le \ell,\ 1 \le c \le r-1$. The Pascal triangle is similar to the ordinary Pascal triangle, but the left and right sides are not filled with ones, but with the given values $u_0, \ldots, u_\ell$ on the left side and $v_0, \ldots, v_\ell$ on the right side; and the recurrence is the usual recurrence of binomial numbers, but the initial conditions are given by $\mathbf{u}$ and $\mathbf{v}$ and the sum is done in F_2. Figure on the right of figure 1 shows a graphic representation of a Pascal triangle.

Fig. 1. Exemples of Steinhaus triangle and Pascal triangle, respectively

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A Steinhaus or Pascal triangle is said to have *rotational symmetry* if its graphical representation is invariant under rotations of 120 and 240 degrees, and it is said to have *dihedral symmetry* if it has rotational symmetry and the graphical representation is invariant by axial symmetry with respect to the height of the triangle. Our goal here is to give formulae for explicitly obtaining the initial sequences that produce Steinhaus and Pascal triangles with rotational and dihedral symmetry.

Hypergraphs for computing Determining Sets of Kneser Graphs *

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Abstract. A set of vertices S is a *determining set* of a graph G if every automorphism of G is uniquely determined by its action on S. The *determining number* of G is the minimum cardinality of a determining set of G. This paper studies determining sets of Kneser graphs from a hypergraph perspective. This new technique lets us compute the determining number of a wide range of Kneser graphs. We also show its usefulness by giving shorter proofs of the characterization of all Kneser graphs with fixed determining number 2, 3 or 4, going even further to fixed determining number 5. We finally establish for which Kneser graphs $K_{n:k}$ the determining number is equal to $n-k$, answering a question posed by Boutin.

Key words: Determining set, determining number, Kneser graph, hypergraph.

1 Introduction

The *determining number* of a graph G, denoted by $\mathrm{Det}(G)$, is the minimum cardinality of a set $S \subset V(G)$ such that the automorphism group of the graph obtained from G by fixing every vertex in S is trivial. The set S is called a *determining set* of G. Although they were first defined as *fixing sets* by Harary [2], we follow the terminology of [1] since the author develops a study on Kneser graphs. The Kneser graph $K_{n:k}$ has vertices associated with the $k-$subsets of the $n-$set $[n] = \{1, \ldots, n\}$ and edges connecting disjoint sets.

This work addresses a general study of determining sets of Kneser graphs. Our main contribution is to introduce hypergraphs as a tool for finding determining sets. This technique lets us compute the determining number of all Kneser graphs $K_{n:k}$ verifying that $n \geq \frac{k(k+1)}{2} + 1$, which is a significant advance since the only exact values obtained previously are for $n = 2^r - 1$ and $k = 2^{r-1} - 1$ (see [1] for details). We also list all Kneser graphs with fixed determining number 2, 3, 4 or 5, showing that hypergraphs play an important role since our technique provides, for instance, direct methods for selecting from a list of 196 candidate Kneser graphs, those which indeed have determining number equal to 5. Finally, we answer the question of whether there

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exists an infinite family of Kneser graphs $K_{n:k}$ with $k \geq 2$ with determining number $n-k$, which was posed by Boutin in [1].

2 Some Main Results

For any set of vertices $S \subseteq V(K_{n:k})$ denote by $\mathcal{H}_S$ the k-regular hypergraph obtained as follows. The vertex set $V(\mathcal{H}_S)$ is equal to S, and two vertices are adjacent whenever they contain a common element. When an element of $[n]$ appears only once in the vertices of S, we have a loop in the corresponding vertex of $\mathcal{H}_S$. We now state some of the main results obtained in this paper.

Lemma 1. *A vertex set S is a determining set of $K_{n:k}$ with $n \geq 2k+1$ if and only if the associated k-regular hypergraph $\mathcal{H}_S$ is simple and has either n or $n-1$ edges.*

Lemma 2. *For any $k-$regular simple hypergraph $\mathcal{H}$ with either n or $n-1$ edges and $n \geq 2k+1$, there exists a determining set S of $K_{n:k}$ such that $\mathcal{H} \cong \mathcal{H}_S$.*

Theorem 1. *Let k and d be two positives integers such that $k \leq d$ and $d > 2$. Then,* $\mathrm{Det}\left(K_{\lfloor \frac{d(k+1)}{2} \rfloor + 1:k}\right) = d$.

Theorem 2. *Let k and d be two positives integers verifying that $3 \leq k+1 \leq d$. For every $n \in \mathbb{N}$ such that $\lfloor \frac{(d-1)(k+1)}{2} \rfloor < n < \lfloor \frac{d(k+1)}{2} \rfloor$ it holds that* $\mathrm{Det}(K_{n+1:k}) = d$.

Proposition 1. *The Kneser graphs with determining number 5 are* $K_{6:1}$, $K_{8:2}$, $K_{10:3}$, $K_{11:3}$, $K_{12:4}$, $K_{13:4}$, $K_{13:5}$, $K_{14:5}$, $K_{15:5}$, $K_{16:5}$, $K_{14:6}$, $K_{15:6}$, $K_{16:6}$, $K_{17:6}$, $K_{16:7}$, $K_{17:7}$, $K_{18:7}$, $K_{19:7}$, $K_{17:8}$, $K_{18:8}$, $K_{19:8}$, $K_{20:8}$, $K_{21:8}$, $K_{19:9}$, $K_{20:9}$, $K_{21:9}$, $K_{22:9}$, $K_{21:10}$, $K_{22:10}$, $K_{23:10}$, $K_{24:10}$, $K_{23:11}$, $K_{24:11}$, $K_{25:11}$, $K_{26:11}$, $K_{25:12}$, $K_{26:12}$, $K_{27:12}$, $K_{27:13}$, $K_{28:13}$, $K_{29:14}$, *and* $K_{31:15}$.

Theorem 3. $\mathrm{Det}(K_{n+1:k}) = n+1-k$ *if and only if $K_{n+1:k}$ is isomorphic to* $K_{n+1:1}$, $K_{5:2}$ *or* $K_{6:2}$.

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Boundary-type sets and product operators in graphs ⋆

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Abstract. We present the description of boundary-type sets, such are the extreme vertex set, the boundary, the eccentricity, the periphery and the contour of strong and lexicographic products of graphs, in terms of factor graphs. Similar descriptions can be found in the literature for the cartesian product of graphs.

The problem of rebuilding the vertex set of a graph using convex-type operations has attracted much attention since it was proved by Farber and Jamison (see [9]) that every convex subset in a graph is the convex hull of its extreme vertices if and only if the graph is chordal and contains no induced 3-fan. In other words, *extreme vertices*, that is, vertices whose neighborhood induces a complete graph, can be used to rebuild the vertex set of a graph just in a particular graph class. So the natural question about which vertex set could play a similar role in general graphs arises, and a number of boundary-type sets can be used to answer it. We focus on the following vertex subsets: extreme vertex set ($\mathrm{Ext}\,(G)$) , boundary ($\partial(G)$, see [4]), periphery ($\mathrm{Per}\,(G)$) , eccentricity ($\mathrm{Ecc}\,(G)$, see [5]) and contour ($\mathrm{Ct}\,(G)$, see [4]).

Remember that the *interval* $I[u,v]$ between two vertices u,v of a graph G is the set of all vertices lying in any shortest path between u and v. Also if $S \subset V(G)$, the interval of S is the set $I[S] = \bigcup_{u,v \in S} I[u,v]$. So S is called a *geodetic set* when $I[S] = V(G)$. On the other hand, $S \subseteq V(G)$ is called *convex* if $I[u,v] \subseteq S$, $\forall u,v \in S$, and the *convex hull* $CH(A)$ of $A \subseteq V(G)$ is the smallest convex set containing it. A *hull set* is $A \subseteq V(G)$ with $\mathrm{CH}(A) = V(G)$

The role of the boundary-type sets in rebuilding operations using both intervals and convex hulls, has been studied in [2,4], showing that $\mathrm{Ct}\,(G)$ is a hull set in any graph and it is also a geodetic set in special graph-classes, such are chordal graphs and distance-hereditary graphs. Also $\mathrm{Ct}\,(G) \bigcup \mathrm{Ecc}\,(\mathrm{Ct}\,(G))$ and $\partial(G)$ are geodetic sets in any graph.

Note that the periphery and the eccentricity are natural boundary-type sets defined using the eccentricity of vertices in a graph, however they do not play an remarkable role in term of rebuilding using convex operations in general graphs. The boundary and the contour arise to this end and, although their definitions are less natural, they have proved useful for this problem. The description of these boundary-

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type sets in cartesian products of graphs can be found in [1] and, in this work, we present similar results in strong products and lexicographic products of graphs.

The *strong product* (see [8]) $G \boxtimes H$ is the graph having $V(G) \times V(H)$ as vertex set and two vertices (g, h) and (g', h') are adjacent if $gg' \in E(G), h = h'$, or $g = g', hh' \in E(H)$, or $gg' \in E(G), hh' \in E(H)$

On the other hand, the *lexicographic product* (see [7]) $G \circ H$ is the graph having $V(G) \times V(H)$ as vertex set and two vertices (g, h) and (g', h') are adjacent if $gg' \in E(G)$, or $g = g', hh' \in E(H)$.

On the case of strong product, extreme vertex set, boundary, periphery, eccentricity and contour can be described in terms of factor graphs and they depend on the relation between diameters and radii of both graphs. On the case of lexicographic product $G \circ H$, all these sets can be also described in terms of factor graphs and there are different cases depending of radii of both graphs and the cardinality of set of vertices with minimum eccentricity of H.

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Geodetic and hull numbers of strong products of graphs *

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Classic convexity can be extended to graphs in a natural way by considering shortest paths, also called geodesics: a set S of vertices of a graph is convex if it contains all the vertices lying in some geodesic with endpoints in S and the convex hull of a set S of vertices is the minimum convex set containing S. Farber and Jamison [9] characterized the graphs such that every convex set is the convex hull of its simplicial vertices (i.e. vertices such that its neighborhood induces a complete graph). This rebuilding problem can be studied for general graphs from different points of view. Geodetic and hull numbers give how many vertices are needed, at least, to rebuild the vertex set of a graph by using the closed interval and the convex hull operations respectively. This problem has been studied for different graph classes obtained by means of graph operations. For example, in cartesian products [1,5], compositions [6] and joins [7] of graphs. In this work, we develop these topics for strong products of graphs.

All the graphs considered are finite, simple and connected graphs. An $x-y$ path of length $d(x,y)$ is called an $x-y$ *geodesic*. The *closed interval* $I[x,y]$ consists of all vertices lying in some $x-y$ geodesic of G. For $S \subseteq V(G)$, the *geodetic closure* $I[S]$ of S is the union of all closed intervals $I[u,v]$ over all pairs $u,v \in S$, i.e. $I[S] = \bigcup_{u,v \in S} I[u,v]$. A set S of vertices of G is *geodetic* if $I[S] = V(G)$ and *convex* if $I[S] = S$. The *convex hull* of $S \subseteq V(G)$ is the smallest convex set containing S and is denoted by $CH(S)$ [8]. A set $S \subseteq V(G)$ is said to be a *hull set* if its convex hull is the whole vertex set $V(G)$. The *geodetic number* and the *hull number* of a graph G are respectively the minimum cardinality among all geodetic sets and hull sets [8,10]. We denote them by $g(G)$ and $h(G)$. Certainly, every geodetic set is a hull set, and hence, $h(G) \leq g(G)$. The geodetic and hull numbers of paths, cycles, complete graphs, trees and many other classes of graphs is well known. The *strong product* $G \boxtimes H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H) = \{(g,h) : g \in V(G), h \in V(H)\}$ in which vertices (g,h) and (g',h') are adjacent whenever (1) $g = g'$ and $hh' \in E(H)$, or (2) $h = h'$ and $gg' \in E(G)$, or (3) $gg' \in E(G)$ and $hh' \in E(H)$.

In this work we first study some relations between the geodetic and hull sets of the strong product $G \boxtimes H$, and the geodetic and hull sets of its factor graphs G and H. We prove that if S_1 is geodetic in G_1 and S_2 is geodetic in H, then $S_1 \times S_2$ is geodetic in $G \boxtimes H$. The same result holds for hull sets. We have also that if S is a

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geodetic set of $G \boxtimes H$, then the projection of S onto G or the projection of S onto H is a geodetic set. This is not true for a hull set of $G \boxtimes H$. These results allows us to find some bounds of the geodetic and hull number of the strong product $G \boxtimes H$ in terms of the geodetic and hull numbers of G and H.Concretely, the geodetic number of $G \boxtimes H$ satisfies $\min\{g(G), h(G)\} \leq g(G \boxtimes H) \leq g(G)g(H)$ and for the hull number we have $h(G \boxtimes H) \leq h(G)h(H)$. Moreover, all these bounds are sharp. Finally we determine the geodetic and hull number of the strong product of paths, cycles and complete graphs.

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Valores extremos en los parámetros de dominación y resolución de un grafo *

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Palabras clave: Conjuntos dominantes, número de dominación, conjuntos que resuelven un grafo, dimensión métrica de un grafo, $\ell - d$-conjuntos.

Sea $G = (V(G), E(G))$ un grafo y $S \subset V(G)$. Decimos que S domina G, o que es un conjunto dominante de G, si se verifica que todo vértice de $V(G) \setminus S$ tiene algún vecino en S. El número de dominación de un grafo G es el mínimo cardinal de un conjunto que domina G y lo notaremos por $\gamma(G)$.

Sean u, v, z tres vértices de un grafo G. Se dice que z resuelve (o localiza) los vértices u y v si $d(u, z) \neq d(v, z)$. Un conjunto de vértices $S \subset V(G)$ se dice que resuelve G si para todo par de vértices de G existe uno de S que los resuelve. Se llama dimensión métrica de G al mínimo cardinal de un conjunto que resuelve G y lo notamos por $\beta(G)$.

A los conjuntos que resuelven y dominan un grafo G los llamaremos η-conjuntos y notamos $\eta(G)$ al mínimo cardinal de uno de estos conjuntos.

Un conjunto dominante S de un grafo G que además verifica que $N(u) \cap S \neq N(v) \cap S$ para todo par $u, v \in V(G) \setminus S$ se dice que es un $\ell - d$-conjunto y notamos $\lambda(G)$ al mínimo cardinal de un $\ell - d$-conjunto de G.

Observamos que si S es $\ell - d$-conjunto entonces S también resuelve y por tanto $\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \lambda(G)$. Por otra parte, la reunión de un conjunto que resuelve y otro que domina es un η-conjunto, de donde $\eta(G) \leq \gamma(G) + \beta(G)$.

Los conjuntos que resuelven y la dimensión métrica de un grafo han sido largamente estudiados (ver [4],[2]). Los $\ell - d$-conjuntos fueron introducidos por P.J. Slater en [9] y los η-conjuntos por M.A. Henning y O. Oellermann en [7]. Estos conjuntos han sido estudiados en diferentes tipos de grafos (ver [1],[3],[6]). Una motivación para el estudio de estos conjuntos es el detectar fallos en la diagnosis de sistemas con multiprocesadores.

En este trabajo estudiamos los valores extremos de los parámetros η y λ.

Primero obtenemos cotas ajustadas de $\eta(G)$ en función del orden y el diámetro del grafo y se caracterizan todos los grafos con $\eta(G)$ igual a 1 ó 2. Después se hace

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un estudio análogo con el parámetro λ. También se estudian algunas condiciones en las cuales $\lambda = \eta$ y se caracteriza todos los grafos con $\lambda(G) = n - 2$ siendo n el orden del grafo. Finalmente verificamos la existencia de grafos con número de dominación, dimensió métrica y parámetro η cualesquiera siempre que se verifiquen las restricciones elementales.

Todos los grafos considerados son finitos, simples y conexos. Para conceptos básicos de teoría de grafos el lector puede referirse a [10].

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A new framework for domination*

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Abstract. Dominating concepts constitute a cornerstone in graph theory. Part of the efforts in the field have been focused in finding different mathematical frameworks where domination notions arise naturally and which provides new points of view about them. In this paper, we introduce one of this frameworks based in convexity. The main idea consists of defining a convexity in a graph, already used in image processing, for which the usual parameters of convexity are closely related to domination parameters. Moreover, the Helly number of this convexity may be viewed as a new domination parameter whose study would be of interest.

Key words: digital convexity, Helly number, domination chain

1 Introduction

A subset of vertices S in a (simple, undirected and connected) graph G is called *dominating* if any vertex in $V(G) \setminus S$ is adjacent to some vertex of S.In Figure 1, the sets $S = \{a, c, e\}$ and $S' = \{a, d, f, h\}$ are dominating sets.

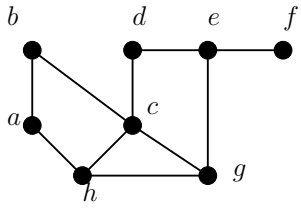

Fig. 1. $\{a, c, e\}$ and $\{a, d, f, h\}$ are dominating sets.

Domination theory plays a central role in the field of Graph Theory deeply connected with cliques and independent sets. There exist around 2400 papers on the topic and the growing rate does not seem to decline. Therefore, there exists an enormous

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interest in finding mathematical frameworks in which domination concepts arise naturally. In [1], the authors showed ten of these mathematical frameworks ranged from hypergraphs to linear and integer programming, and they pointed out that:

"Each framework provides a unifying theory and a generalized viewpoint that enables one to identify and define new parameters, see relationships among these parameters and develop insight the computational problems involving these parameters."

They also suggested that:

"...this is but the beginning, that is, other frameworks for the domination number undoubtedly exists, each of which provides a rich understanding of the concept of domination in graphs."

In this paper, we introduce a new framework for domination based in defining a special graph convexity. This convexity has been largely used in the field of image processing [2], mainly for filtering digital images [3].

Given a graph G, the *digital convex hull* operator, or *d-convex hull* $CH_d(S)$ of a set of vertices S, is defined as $CH_d(S) = \{u \in V(G) \colon N[u] \subseteq N[S]\}$. This operator defines a convexity in the graph, its *digital convexity*, whose convex sets are called *d-convex* (for details about graph convexities see [4]).

We proved that some parameters that usually describe a convexity, when particularized for the the digital convexity, can be reinterpreted in terms of domination. Specifically, the *convex number* of a graph is an upper bound of all the domination parameters, the *upper* and *lower rank* agree with the *upper* and *lower irredundance numbers* of the graph, and finally the *upper Helly number* is the *upper domination number* of G.

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Supergeodeticidad en grafos. ⋆

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Resumen. Dado un grafo conexo $G = (V, E)$, la distancia $d(u, v)$ entre dos vértices $u, v \in V$, es la longitud del camino más corto uniendo u y v en G. Un camino $u-v$ de longitud $d(u, v)$ se llama $u-v$ *geodésica*. Definimos entonces $I[u, v]$ como el conjunto de todos los vértices que están en alguna $u - v$ geodésica. Dado $S \subset V$ llamamos *clausura geodética* de S, y lo denotamos $I[S]$, al conjunto de vértices que están en alguna geodésica entre dos vértices cualesquiera de S. Es decir,

$$I[S] = \bigcup_{u,v \in S} I(u, v)$$

Si $I[S] = S$, se dice que S es un *conjunto convexo* de G y si $I[S] = V$, decimos que S es un *conjunto geodético*. Llamamos *número geodético*, y lo denotamos $g(G)$, al cardinal de un conjunto geodético mínimo. El problema de encontrar conjuntos geodéticos de cardinal mínimo ha sido ampliamente estudiado [5,2,3] puesto que nos permiten reconstruir un grafo considerando los caminos mínimos entre los vértices de esos conjuntos.

En este trabajo introducimos un nuevo invariante para un grafo a partir del concepto de geodeticidad. Diremos que $S \subset V$ es un *conjunto supergeodético* si existe un vértice destacado $u \in S$ tal que todos los vértices de G están en alguna geodésica entre u y cualquier otro vértice de S, es decir, S es supergeodético si $\exists u \in S$ tal que

$$\bigcup_{v \in S} I(u, v) = V$$

De igual forma llamamos *número supergeodético*, y lo denotaremos $sg(G)$, al cardinal de un conjunto supergeodético mínimo. En la Figura 1 podemos ver un ejemplo de un grafo cuyo número geodético es 3, siendo $S = \{u, v, w\}$ el único conjunto geodético mínimo, mientras que su número supergeodético es 4, con $S' = \{u, v, w, x\}$ un conjunto supergeodético, con u el vértice destacado.

Puesto que en los conjuntos supergeodéticos tenemos todos los caminos mínimos sólo considerando uno de los vértices, el coste de reconstruir un conjunto convexo en un grafo se reduce mucho si el conjunto de partida es supergeodético y no sólo geodético.

Como podemos deducir de las definiciones, claramente $g(G) \leq sg(G)$. En la Sección 1 estudiaremos esta cota y veremos que es posible encontrar grafos en los que

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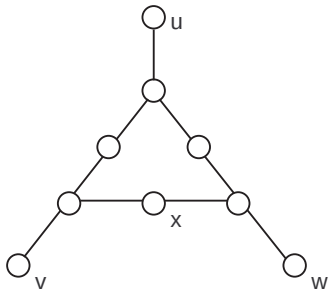

Figura 1. Un grafo con número geodético 3 y supergeodético 4.

la diferencia entre los números geodético y supergeodético es tan grande como se quiera. En esta sección además, daremos otros teoremas de realizabilidad relacionando el número supergeodético no sólo con el número geodético sino también con otros invariantes de un grafo como el orden, el diámetro y el número geodético de Steiner.

Si bien los conjuntos destacados de vértices de un grafo habitualmente geodéticos no son supergeodéticos en general, en la Sección 3 comprobaremos que para ciertas familias de grafos esta afirmación sí es cierta, como por ejemplo en grafos intervalo.

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Local spectrum of the subconstituents and completely pseudo-regular codes *

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Abstract. The local spectrum of a set has been proven to be of great utility to study several metric properties of sets of vertices. It also has applications in the area of pseudo-distance-regularity around a set and can be used to obtain quasi-spectral characterizations of completely (pseudo-)regular codes. In this paper we study the relation between the local spectrum of a set and that of its subconstituents. Moreover, we obtain a characterization for completely pseudo-regular codes, and consequently for completely regular codes, in terms of the relation between the local spectrum of an extremal set and the local spectrum of its antipodal set.

Key words: Completely regular codes, subconstituents, local spectrum, pseudo-distance-regularity.

The notion of local spectrum was first introduced, for a single vertex of a graph, in [6]. In the study of pseudo-distance-regularity around a set of vertices, which is closely related to completely regular codes, in [5] the local spectrum is generalized to sets of vertices. In this work we are interested in the study of the relation between the local spectrum of a set and that of the subconstituents associated to it. Given a set of vertices C in a simple connected graph we denote the k-th subconstituent associated to it by C_k, $k = 0, 1, \ldots, \varepsilon_C$. That is, C_k is the set of vertices at distance k from C. As some results point out [3,5], the set of vertices at maximum distance from C will play an important role, we will refer to this set as the antipodal set of C and if there is no possible confusion we will write $D = C_{\varepsilon_C}$. We proof that, if C is an extremal set, the C and D multiplicities (see [4]) are highly related, and, consequently, the C-local and D-local spectrum also.

Proposition 1. *Let C be an extremal set and let D be its antipodal set. Then,* $\mathrm{ev}_C\,\Gamma \subset \mathrm{ev}_D\,\Gamma$ *and the C-multiplicities and D-multiplicities satisfy*

$$m_C(\mu_l)m_D(\mu_l) \geq \frac{\pi_0^2(C)}{\pi_l^2(C)}\,\frac{\|\boldsymbol{\rho}C\|^2\|\boldsymbol{\rho}D\|^2}{\|\boldsymbol{\nu}\|^4} \qquad \text{for all } \mu_l \in \mathrm{ev}_C\,\Gamma,$$

where equality is equivalent to the linear dependence of the projections of $\boldsymbol{\rho}C$ and $\boldsymbol{\rho}D$ the eigenspace of μ_l.

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What is more, if we have equality in Proposition 1 for all the eigenvalues of the C-local spectrum then there exist a polynomial relating D and C. That suggest that the existence of C-local distance polynomials, implying that C is a completely pseudo-regular code, can guarantee a tight relation between the local spectrum of the subconstituents. Next result support this claim.

Proposition 2. *Let C be a completely pseudo-regular code in Γ. Then $\mathrm{ev}_{C_k}\Gamma \subset \mathrm{ev}_C\Gamma$. Moreover, the orthogonal projections of $\boldsymbol{\rho}C_k$ and $\boldsymbol{\rho}C$ onto each eigenspace are linearly dependent.*

Finally, as a kind of counterpart we obtain a new characterization for completely pseudo-regular codes, and, consequently, for a completely regular code.

Proposition 3. *Let C be an extremal set with eccentricity ε. Then C is a completely pseudo-regular code if and only the orthogonal projections of $\boldsymbol{\rho}C$ and $\boldsymbol{\rho}D$ onto each eigenspace of Γ are colinear.*

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Diseño de redes altamente sincronizables y robustas

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1 Resumen

Las redes complejas nos rodean. Estan formadas por un gran conjunto de nodos en continua evolución que representan las entidades del sistema. Una de las propiedades más interesantes de las redes es la sincronización de todos sus nodos, debido además a sus múltiples aplicaciones en el mundo de las redes de comunicación. Cuando se estudia esta propiedad, se consideran n osciladores idénticos conectados por enlaces iguales y bidireccionales. Por lo tanto, su representación matemática es un grafo simple. La evolución temporal de cada i-ésimo oscilador viene dada una ecuación basada en la matriz laplaciana asociada al sistema, es decir, $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$, donde $\boldsymbol{D}$ una matriz diagonal con los grados de los vértices y $\boldsymbol{A}$ es la matriz de adyacencia del grafo que modela la red.

Es un hecho conocido que la red presenta una buena sincronización, si la razón $Q = \frac{\mu_1}{\mu_{n-1}}$ es lo más pequeña posible, donde $\mu_1 \geq \cdots \geq \mu_{n-1} > 0$ son los autovalores de la matriz laplaciana. Además, el autovalor μ_{n-1} recibe el nombre de conectividad algebraica del grafo, y está íntimamente relacionado con la robustez de la red y su tolerancia a fallos.

Por consiguiente, la topología de la red es un factor muy importante a la hora de estudiar su sincronización, y en particular el estudio de su espectro.

Asi pues, la razón $R = (\lambda_1 - \lambda_2)/(\lambda_2 - \lambda_d)$, donde $\lambda_1 > \ldots > \lambda_d$ son los d distintos autovalores de la matriz de adyacencia del grafo, se utiliza como una medida de la distancia entre el primer autovalor y la mayor parte del la concentración de autovalores. Por diversos motivos, conviene que las razones:

$$\omega_1(G) = \frac{1}{R} \quad y \quad \omega_2(G) = R + 1, \tag{1}$$

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sean pequeñas. Se puede comprobar fácilmente que son iguales cuando las dos coinciden con el número áureo φ. Lo cual nos conduce a la definición de los grafos que vamos a estudiar.

Definición 1. *Un grafo G tiene espectro áureo (es un GSG) si*

$$\omega_1(G) = \omega_2(G) = \varphi,$$

donde $\varphi = (1 + \sqrt{5})/2$ es el número áureo o razón de oro.

Nuestros resultados nos conducen a construir analíticamente nuevos grafos GSG a partir de operaciones clásicas en la teoría de grafos, como el producto directo, o a utilizar diversas caracterizaciones para encontrar GSG bipartidos regulares de forma computacional, los cuales presentan unas propiedades muy deseadas: buena sincronización, buena expansión, propiedad de ser pequeño mundo, etc, todo ello utilizando resultados conocidos de la teoría espectral de grafos.

También realizamos un estudio de diferentes parámetros topológicos de grafos GSG, como el diámetro, la distancia media, el número cromático, el número isoperimétrico (relacionado con las propiedades de expansión del grafo), el grado máximo, etc.

Dureza de algunas familias de grafos ⋆

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Resumen

Existen diferentes parámetros para medir la vulnerabilidad de una red. Los más conocidos y estudiados son los índices de conectividad y arista-conectividad, los cuales cuantifican el mínimo coste para desconectar una red. Sin embargo, estos parámetros no ofrecen información alguna sobre lo que queda tras la ruptura de la misma. Con el fin de medir de una forma más adecuada la vulnerabilidad de redes, se han introducido y estudiado otros parámetros, como la dureza, la integridad, la tenacidad o el número de dispersión, así como algunas variantes de la conectividad y la arista-conectividad, por ejemplo, la conectividad condicional. Éstos parámetros ya no sólo miden la dificultad de romper una red, sino también proporcionan una idea del daño causado en ella, lo cual resulta de gran interés cuando se pretende reconstruir y reparar la red. Nosotros hemos trabajado con la noción de *dureza*, introducida por Chvátal [3], con la que se relacionan el cardinal de un conjunto que rompe la red con el número de componentes conexas resultantes. Este parámetro se define como

$$\tau(G) = \text{mín}\left\{|S|/\omega(G-S) : S \subseteq J(G)\right\}, \text{ donde}$$

$J(G) = \{S \subset V(G) : S$ es un conjunto de corte o $G - S$ es un vértice aislado$\}$, y $\omega(G-S)$ representa el número de componentes conexas del grafo resultante $G-S$ al eliminar S. Muchos trabajos han prestado especial atención a la idea de relacionar condiciones de dureza con la existencia de estructuras cíclicas [1,2,4]. Sin embargo, el valor exacto de $\tau(G)$ sólo se conoce para unas pocas familias de grafos; para los caminos y los ciclos [7], para el producto cartesiano de dos grafos completos [3] y de caminos y/o ciclos [6], y para la composición de dos grafos, cuando uno de ellos es un camino, un ciclo o un bipartito completo [6].

Nosotros nos centramos en el estudio de la dureza de dos familias de grafos: la corona $G \circ H$ de dos grafos G y H cualesquiera [5] y el producto cartesiano $K_2 \times G$, para cualquier grafo G arbitrario. El grafo corona de dos grafos cualesquiera G y H, denotado por $G \circ H$, es el grafo que se obtiene de tomar una copia de G, $|V(G)|$ copias de H y unir mediante aristas el $i-$ésimo vértice de G con cada uno de los vértices de la $i-$ésima copia de H. El *producto cartesiano* $K_2 \times G$ del grafo completo K_2 por un grafo arbitrario G, denotado por $K_2 \times G$, es el grafo cuyo conjunto de vértices es

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$V(K_2) \times V(G)$ en el que un vértice (i, u), con $i = 1, 2$, es adyacente a otro vértice (j, v), con $j = 1, 2$, si $i = j$ y $uv \in E(G)$, o $i \neq j$ y $u = v$.

Nuestro objetivo en este trabajo es determinar la dureza de la corona $G \circ H$ de dos grafos conexos G y H a partir de parámetros conocidos de éstos. Como consecuencia, obtendremos el valor exacto de la dureza de la corona de dos grafos pertenecientes a algunas familias conocidas, como son las estrellas, los caminos, los ciclos, las ruedas y los grafos completos. Completaremos el trabajo con la prueba de dos acotaciones para la dureza del producto cartesiano $K_2 \times G$, siendo G un grafo cualquiera.

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Algorithmic Method to Determine Filiform Lie Algebras

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1 Introduction

At present, several aspects about Lie algebras remain unknown. In fact, the classification of filiform Lie algebras is still an open problem. These algebras constitute a particular type of nilpotent Lie algebras, introduced by Vergne [3] in the late 1960s.

In a recent paper [2], one of the authors studied certain families of n-dimensional complex filiform Lie algebras by analyzing the invariants z_1 and z_2 previously introduced in [1]. In this paper, we carry on the study done in [2], by introducing an algorithmic method to obtain laws of filiform Lie algebras starting from the maximum among the dimension of their abelian ideals. Note that the results so obtained can be considered a new step forward in solving the open problem of classifying these algebras.

2 Main Results

Proposition 1. *Let $\mathfrak{g}$ be an n-dimensional non-model filiform Lie algebra. Then, $\mathcal{C}^{n-z_2+1}(\mathfrak{g})$ is the unique abelian ideal of maximal dimension. Consequently, $I(\mathfrak{g}) = z_2 - 1$.*

Proposition 2. *Let $\mathfrak{g}$ be an n-dimensional non-model filiform Lie algebra. Then, the law of $\mathfrak{g}$ is given by the following brackets*
$[e_1, e_h] = e_{h-1}$, $3 \leq h \leq n$; $\quad [e_{z_1}, e_{z_2+1}] = \alpha_1 e_2$; $\quad [e_{z_1+1}, e_{z_2+1}] = \alpha_1 e_3 + \alpha_2 e_2$;
$\vdots$
$[e_{z_2}, e_{z_2+1}] = \alpha_1 e_{z_2-z_1+2} + \alpha_2 e_{z_2-z_1+1} + \ldots + \alpha_{z_2-z_1} e_3 + \alpha_{z_2-z_1+1} e_2$;
$[e_{z_1}, e_{z_2+k}] = \alpha_1 e_{k+1} + \alpha_2^1 e_k + \ldots + \alpha_k^{k-1} e_2$;
$[e_{z_1+p}, e_{z_2+k}] = \sum_{h=2}^{k+p} P_h([e_{z_1+p-1}, e_{z_2+k}] + [e_{z_1+p}, e_{z_2+k-1}]) e_{h+1} + \alpha_{k+p}^{k-1} e_2$,
where $2 \leq k \leq n - z_2$, $p < z_2 - z_1 + k$ and P_l (for $1 \leq l \leq n$) is the following function

$$P_l : \mathfrak{g} \to \mathbb{C} : \; u \mapsto P_l(u) := \text{coordinate of } u \text{ with respect to the basis vector } e_l.$$

Corollary 1. *Let* $\mathfrak{g}$ *be an* n*-dimensional non-model filiform Lie algebra with* $I(\mathfrak{g}) = k \in \mathbb{N}$*. Then,* $z_2 = k + 1$ *and the following relation holds:* $3 \leq z_1 - 1 \leq k < n - 1 \leq 2k - 1$.

Now, we show an algorithm to compute the law of an n-dimensional non-model filiform Lie algebra $\mathfrak{g}$ starting from the value of $I(\mathfrak{g})$.

Input

1. The dimension n of a non-model filiform Lie algebra $\mathfrak{g}$.
2. The value k of the invariant $I(\mathfrak{g})$.

Output

1. A list with the triples (z_1, z_2, n) such that there exist non-model filiform Lie algebras associated with these triples.
2. The law of each filiform Lie algebra for each triple.

Method

1. Computing the value of the invariant z_2 by using Proposition 1. According to this value, several possibilities appear for the invariant z_1.
2. For each value of z_1, computing all the possible non-zero brackets of $\mathfrak{g}$ given by Proposition 2.
3. Ruling out those values of z_1 not satisfying the Jacobi identities $J(e_h, e_k, e_l) = 0$, for $z_1 \leq h < k < l \leq n$.
4. Obtaining a list with all the triples (z_1, z_2, n) such that there exist non-model filiform Lie algebras having such invariants.
5. By using Proposition 2 again, computing the law of each algebra associated with a triple given in the previous step.

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Realización geométrica de 2-complejos cebolla

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Resumen. $K = (V(K), E(K), T(K))$ es un 2-*complejo simplicial* donde $V(K)$ es un conjunto no vacío de vértices, $E(K)$ es un conjunto de pares no ordenados (aristas) de elementos de $V(K)$ y $T(K)$ es un conjunto de tripletas no ordenadas (triángulos) de elementos de $E(K)$. Un 2-complejo tiene una *realización geométrica* si admite una inmersión en el espacio, $\Re^3$ tal que todas sus aristas son segmentos rectilíneos y sus triángulos planos. En este trabajo presentamos la realización geométrica de una familia de 2-complejos estratificados por capas, que llamamos 2-*complejos cebolla regulares*.

Palabras clave: geometric realization, 2-complex, triangulation.

1 Introducción

Partiendo de que "No todo 2−complejo que admite una inmersión en el espacio, tiene una realización geométrica, es decir, todas las aristas son segmentos rectilíneos, todos los triángulos son planos y éstos sólo tienen en común las aristas"[1], presentamos una familia de 2-complejos que sí admite realización geométrica en $\Re^3$.

Recordemos que un **2-complejo simplicial** es una tripleta de conjuntos $K = (V(K), E(K), T(K))$ donde $V(K)$ es un conjunto no vacío, $E(K)$ es un conjunto de pares no ordenados de elementos de $V(K)$ y $T(K)$ es un conjunto de tripletas no ordenadas de elementos de $E(K)$. A los elementos de $V(K)$, $E(K)$ y $T(K)$ los llamaremos, respectivamente, vértices, aristas y triángulos. Un **2-complejo simplicial puro**, K, es un complejo simplicial dónde cada símplice es cara de algún 2-símplice. Y se dice que K es **conexo** si existe un 2-camino (sucesión de triángulos adyacentes por las aristas) para cada par de aristas de $E(K)$. Sea $L = (V(L), E(L), T(L))$ un 2-complejo simplicial. Se dice que L es un **subcomplejo** de K si se verifica $V(L) \subseteq V(K)$, $E(L) \subseteq E(K)$ y $T(L) \subseteq T(K)$. Se dice que L es un **subcomplejo recubridor** de K si $V(L) = V(K)$. Un 2-complejo, K, es espacial si admite una inmersión en $\Re^3$, es decir, si existe una aplicación continua $i : K \longrightarrow \Re^3$ de manera que K e $i(K)$ son homeomorfos. Se dice que un 2-complejo espacial tiene una **realización geométrica** si admite una inmersión en el espacio, $\Re^3$, tal que todas sus aristas son segmentos rectilíneos y sus triángulos planos.

2 Nuestro resultado

La aportación de este trabajo es el estudio de una familia de 2-complejos, que sujetos a una serie de condiciones admitirán realización geométrica en $\Re^3$.

Un 2-complejo $K = (V(K), E(K), T(K))$ puro, conexo se dice **estratificado por la n-upla ordenada de subcomplejos** $(K_1, ..., K_n)$ si verifica:

- $V(K) = V_1 \sqcup V_2 \sqcup ... \sqcup V_n$
- Los subcomplejos $K_i \subset K$ generados por los vértices $V_i (i = 1, 2, ..., n)$, respectivamente, son triangulaciones homeomorfas a la 2-esfera S^2.
- Toda arista $e \in E(K - [K_1 \sqcup K_2 \sqcup ... \sqcup K_n])$, es de la forma $e =< v_i, v_{i+1} >$ con $v_i \in K_i$ y $v_{i+1} \in K_{i+1}$.

A $K_1, K_2, ..., K_n$ se le llama **estratificación** de K.

Una inmersión topológica $\varphi : K \longrightarrow \Re^3$ se dice una **inmersión cebolla** respecto a $(K_1, ..., K_n)$ si $\forall i$ $\varphi(K_i)$ está contenido en la componente acotada de $\Re^3 - \varphi(K_{i+1})$.

Un **2-complejo cebolla** en $\Re^3$, (K, φ), es la imagen de un 2-complejo estratificado K por $(K_1, ..., K_n)$ mediante una inmersión cebolla respecto a $(K_1, ..., K_n)$.

Sea $K = (V(K), E(K), T(K))$ un 2-complejo cebolla estratificado por $(K_1, ..., K_n)$. Se dice que es **regular** si $\forall t = (v_1 v_2 v_3)$ triángulo de $T(K_h)$, existe un único $t' = (w_1 w_2 w_3)$ triángulo de $T(K_{h+1})$, de manera que existen seis aristas $e =< v_i, w_j > \in E(K - [K_1 \sqcup K_2 \sqcup ... \sqcup K_n])$ y recíprocamente.

Es decir, un 2-complejo cebolla estratificado regular es aquél en el que todos los estratos tienen el mismo número de triángulos y existe una biyección entre los triángulos de dos estratos consecutivos de forma que cada pareja de dichos triángulos, t y t', constituye una prisma de base y caras trianguladas.

Teorema: Todo 2-complejo cebolla regular, admite realización geométrica en $\Re^3$.

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On the nonexistence of almost Moore digraphs of diameter four ⋆

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Abstract. *Almost Moore* (d,k)*-digraphs* appear in the context of the *degree/diameter problem* as a class of extremal directed graphs, in the sense that their order is one less than the unattainable Moore bound $M(d,k) = 1+d+\cdots+d^k$, where $d>1$ and $k>1$ denote the maximum out-degree and diameter, respectively. So far, the problem of their existence has only been solved when $d=2,3$ or $k=2,3$. Thus, fixing the degree, Miller and Fris [5] proved that the $(2,k)$-digraphs do not exist for values of $k>2$ and, subsequently, Baskoro et al. [1] established the nonexistence of $(3,k)$-digraphs unless $k=2$. On the other hand, Fiol et al. [3] showed that the $(d,2)$-digraphs do exist for any degree. The digraph constructed is the line digraph $L\,K_{d+1}$ of the complete digraph K_{d+1}. Concerning the enumeration of $(d,2)$-digraphs it is known that there are exactly three non isomorphic $(2,2)$-digraphs (see [6]). The classification of $(d,2)$-digraphs was completed in [4] by proving that $L\,K_{d+1}$ is the unique solution, if $d\geq 3$. Moreover, the authors proved the nonexistence of $(d,3)$-digraphs in [2].

In this paper we deal with the case of almost Moore digraphs of diameter $k=4$. Their construction turns out to be equivalent to the search of binary matrices A fulfilling that $AJ = dJ$ and $I + A + A^2 + A^3 + A^4 = J + P$, where J denotes the all-one matrix and P is a *permutation matrix*. Since the eigenvalues of P are roots of unity, the factorization in $\mathbb{Q}[x]$ of the characteristic polynomial of A involves the polynomials $F_n(x) = \Phi_n(1+x+x^2+x^3+x^4)$, where $\Phi_n(x)$ denotes the nth cyclotomic polynomial. More precisely, if $F_n(x)$ is irreducible in $\mathbb{Q}[x]$ then it is a factor of $\det(xI-A)$ and its multiplicity only depends on the cycle structure of P. We conjecture that $F_n(x)$ is always irreducible in $\mathbb{Q}[x]$, unless $n=1,3,6$. Under this assumption, we show how to derive the nonexistence of almost Moore digraphs of diameter four. Right now, by using tools from algebraic number theory, we have been able to prove that $F_n(x)$ for $n\neq 1,3,6$ is either irreducible or factorizes into two irreducible factors of degree $2\varphi(n)$.

Key words: Almost Moore digraph; Cyclotomic polynomial.

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Integridad local de la composición y la corona de dos grafos ⋆

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Palabras clave: Vulnerabilidad de redes, integridad local, composición, corona.

Resumen

Entre los distintos parámetros considerados para controlar la vulnerabilidad de una red, Barefoot, Entringer y Swart [1,2] introducen la *integridad* como una medida que permite establecer el comportamiento de la componente de mayor orden ante la eliminación de un conjunto de nodos en la red. En 1996, sobre una red de espionaje, Cozzens y Wu [3,4] introducen un nuevo parámetro denominado *integridad local*. Si en una red de espionaje, donde cada agente representa un nodo y cada comunicación directa entre agentes una arista, un espía es descubierto la agencia no puede confiar en los agentes que se relacionaban con él. Por tanto, una manera de medir la solidez de este tipo de redes es analizar la resistencia de la red al ser eliminado un conjunto de nodos y sus vecinos.

Cozzens y Wu [3] introducen el concepto de *estrategia de subversión de vértices* sobre un grafo G como el resultado de eliminar sobre el grafo el entorno cerrado $N[S]$ de un conjunto de vértices S del grafo. El conjunto de vértices S es denominado *conjunto subversivo* y el grafo resultante de la estrategia de subversión es denotado por G/S, observése que $G/S = G\backslash N[S]$. La *integridad local* de un grafo G, denotada por $NI(G)$, se define como:

$$NI(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\},$$

donde $\omega(H)$ es el orden de la mayor componente conexa en el grafo H.

Del valor de la integridad local de un grafo, al igual que de otros parámetros de vulnerabilidad [2] — *dureza*, *tenacidad*, *número de dispersión*—, se conocen sólo algunos resultados parciales: en árboles, ciclos y potencias de ciclos [3,4], en grafos totales [7], en las familias $K_2 \times P_n$, $K_2 \times C_n$ y $W_{1,n} \circ H$ [6], y en la composición de dos caminos $P_n[P_m]$ o de un camino y un ciclo $P_n[C_m]$ [8,9]. Por otra parte, Gambrell [5] acota el valor de la integridad local de los hipercubos y conjetura que

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la integridad local de un grafo G siempre está acotada por $\lceil |V(G)|/3 \rceil$. Kirlangic [6] acota inferiormente el caso general para la corona.

Nuestro trabajo se centra en el estudio del comportamiento de la integridad local sobre dos operaciones binarias sobre grafos: la composición y la corona de dos grafos.

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Triangulaciones y Pseudotriangulaciones de Peso Mínimo: resolución aproximada con *Simulated Annealing* ⋆

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Palabras clave: Geometría Computacional, Triangulaciones, Pseudotriangulaciones, Triangulación de Peso Mínimo, Pseudotriangulación de Peso Mínimo, Metaheurísticas, Simulated Annealing.

1 Resumen

Muchos problemas de optimización en configuraciones geométricas son NP-duros. En este artículo, consideramos los problemas de Triangulación de Peso Mínimo (*Minimum Weight Triangulation, MWT*) y Pseudotriangulación de Peso Mínimo (*Minimum Weight Pseudo-triangulation, MWPT*) para un conjunto dado de puntos en el plano, y mostramos cómo la técnica metaheurística Recocido Simulado (*Simulated Annealing, SA*) permite resolverlos de forma aproximada. Además, mostramos el plan que se llevará a cabo en el estudio experimental para comprobar el desempeño de dicha técnica en los problemas mencionados MWT y MWPT.

El problema de minimizar la longitud total de una triangulación o una pseudotriangulación, es decir, la suma de las longitudes de las aristas, nos permite obtener una medida de calidad para determinar cuán buena es la estructura geométrica, siendo uno de los problemas de optimización más estudiados en el ámbito de la Geometría Computacional.

Mulzer y Rote [10] demostraron en 2006 que el problema MWT es de naturaleza NP-dura, eliminando dicho problema de la lista de problemas abiertos presentado por Garey y Johnson [6]. La complejidad para el problema MWPT es aún desconocido, aunque Levcopoulos y Gudmundsson [7] indicaron cómo obtener una 12-aproximación

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de una pseudotriangulación de peso mínimo, que puede ser calculada en tiempo $O(n^3)$. Además, obtuvieron una $O(\log n \cdot w(MST))$ aproximación de una pseudotriangulación de peso mínimo, calculada en tiempo $O(n \log n)$, donde $w(MST)$ es el peso del árbol generador mínimo.

Dada la dificultad inherente de dichos problemas, los algoritmos aproximados surgen como candidatos alternativos para su aplicación. Éstos pueden dar soluciones cercanas a las óptimas y pueden ser específicos para un problema tratado o formar parte de una estrategia general aplicable en la resolución de distintos problemas, como lo son las técnicas metaheurísticas.

Considerando el actual estado del arte de los problemas presentados hemos determinado resolverlos utilizando técnicas metaheurísticas, ya que son los métodos más apropiados para encontrar soluciones aproximadas a las óptimas. En [2,4,5,3] se puede observar el trabajo realizado por los autores, sobre los problemas MWT y MWPT utilizando metaheurísticas, donde se describe el diseño de algoritmos y resultados experimentales obtenidos utilizando la técnica ACO.

En particular, en esta etapa del tratamiento de los problemas MWT y MWPT, estudiamos y aplicamos la técnica Recocido Simulado. Esta estrategia se basa en la analogía con el proceso de templado o enfriado controlado en la elaboración de ciertas sustancias como vidrio o aceros y el problema de resolver grandes problemas de optimización combinatoria [9,8,1]. Para realizar dicha estrategia, SA introduce un parámetro de control T, designado por la *temperatura*, cuyo valor inicial debe ser alto y se debe decrementar durante el proceso de búsqueda. En dicho proceso se realiza la ejecución de varias iteraciones del algoritmo hasta que se alcanza la condición de finalización. El parámetro de control T permite, con cierta probabilidad, moverse a soluciones y del espacio de búsqueda S cuyos valores de función objetivo $f(y)$ son peores que el valor de función objetivo $f(x)$ de la solución actual $x \in S$. Estos movimientos se llaman ascendentes (*uphill moves*), y la probabilidad se denomina *función de aceptación*.

En este artículo presentamos el diseño de algoritmos SA para la obtención de soluciones aproximadas a los problemas MWT y MWPT, mostrando cuál será el escenario paramétrico con el cual se iniciará la etapa experimental para luego poder verificar con qué valores de parámetros se obtienen mejores soluciones. Presentamos los aspectos teóricos y estado del arte de los problemas planteados; describimos los parámetros y los operadores de vecindad, que se utilizan en la evaluación experimental, y, finalmente, damos nuestras conclusiones y visión de futuro de esta investigación.

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Asymptotic study of subcritical graph classes

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In this work we present a general framework to enumerate in a unified way a wide variety of labelled and unlabelled classes of graphs. Our main contribution is a universal method for rich classes of *unlabelled* graphs, the so-called "subcritical" classes of graphs, which is established through the novel singularity analysis of counting series and yields asymptotic estimates and limit laws for various graph parameters.

Another main contribution is asymptotic estimates and limit laws for various graph parameters of unlabelled series-parallel graphs. We study the class of unlabelled series-parallel graphs, firstly as an important subclass of planar graphs whose asymptotic study has not been carried out so far and therefore it is interesting in its own right, and secondly as a concrete prototype-example to illustrate how our general method is applied. A graph is *series-parallel* (SP-graphs for conciseness) if its 2-connected components are obtained from a single edge by recursive subdivision of edges (series operation) and duplication of edges (parallel operation). Equivalently, SP-graphs can be defined in terms of minors as graphs which exclude K_4 as a minor. Finally, a graph is series-parallel if and only if its tree-width is at most 2. Applying our general method, we show that the number g_n of unlabelled SP-graphs on n vertices is asymptotically of the form

$$g_n \;=\; c\, n^{-5/2}\, \rho^{-n}\, (1 + o(1)),$$

where $\rho \approx 0.10655$. Let G_n be a graph chosen uniformly at random among all unlabelled SP-graphs on n vertices. The random variable X_n counting the number of edges (blocks, or cut-vertices) in G_n features a central limit law

$$\frac{X_n - \mathbb{E}\, X_n}{\sqrt{\mathbb{V}\mathrm{ar}\, X_n}} \to N(0,1),$$

where $\mathbb{E}\, X_n = \mu n + O(1)$ and $\mathbb{V}\mathrm{ar}\, X_n = \sigma^2 n + O(1)$ for computable constants μ and σ^2. In addition, the random variable X_n^k counting the number of vertices of degree k (for k fixed) in G_n satisfies a central limit law with mean $\mathbb{E}\, X_n^k = \mu_k n + O(1)$ and variance $\mathbb{V}\mathrm{ar}\, X_n^k = \sigma_k^2 n + O(1)$ where μ_k and σ_k^2 are computable constants.

Furthermore we show that the same subexponential term $n^{-5/2}$ appears in other classes of graphs (in both labelled and unlabelled cases), which is, roughly speaking, inherited from a tree-like nature. This behaviour appears as a consequence of

a *subcritical composition* scheme which appears in the specification of the counting series associated to connected graphs of the class. Such classes of graphs arising from subcritical composition scheme are called *subcritical* classes of graphs.

We consider block-stable classes of graphs: we say a class $\mathcal{G}$ of graphs *block-stable* if and only if for each graph $G \in \mathcal{G}$, each of its 2-connected components (also called *blocks*) belongs to $\mathcal{G}$. The class of cacti graphs, the class of outerplanar graphs, the class of SP-graphs, and other classes of graphs defined in terms of a class of 2-connected components are block-stable. Additionally, we consider classes of graphs which are defined by a *finite* set of 3-connected graphs. Observe that SP-graphs can be seen as graphs without 3-connected components. We show that the classes of cacti graphs, outerplanar graphs, and SP-graphs, are subcritical and prove that the asymptotic estimates and limit laws for graph parameters of subcritical classes of graphs follow the same asymptotic pattern and limit laws as the class of SP-graphs.

The asymptotic study of subcritical classes of graphs consists of two steps: formal and analytic steps. The formal step consists in translating Tutte's seminal ideas on decomposing graphs into components of higher connectivityin terms of the *decomposition grammar*, which is comparable to the ones introduced by Chapuy, Fusy, Kang and Shoilekova. This decomposition grammar translates combinatorial conditions into functional equations satisfied by the counting series of various classes of graphs. These counting series depend on the connectivity degree and the way how the graphs are rooted. In analytic step, we extract *singular expansions* of the counting series from the systems of functional equations. The main ingredient in this step is from the work of Drmota on system of functional equations, in which precise singular expansions are deduced for very general systems of functional equations. Finally, we derive asymptotic formulas from these singular expansions by extracting coefficients, based on the transfer theorems of singularity analysis developed by Flajolet and Sedgewick.

On the Feng-Rao numbers

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Abstract. Some computations concerning the Feng-Rao numbers are presented. The main results are addressed to numerical semigroups with embedding dimension two.

Key words: Numerical semigroups, generalized order bound, Feng-Rao numbers.

1 Introduction

Let S be a numerical semigroup with genus g and conductor c. It is well-known that $c \leq 2g$, and the semigroup S is called *symmetric* provided $c = 2g$. A very classical case of symmetric semigroup consists of those generated by two elements, i.e. with embedding dimension two (see [4]). On the other hand, we can enumerate the elements of S in increasing order as

$$S = \{0 = \rho_1 < \rho_2 < \rho_3 < \cdots\},$$

In this situation, for $m_1 \in S$, let $\mathrm{D}(m_1) := \{p \in S \mid m_1 - p \in S\}$ and let $\nu(m_1) := \sharp\mathrm{D}(m_1)$. The (classical) *Feng-Rao distance* of S is defined by $\delta_{FR}(m) := \min\{\nu(m_1) \mid m_1 \geq m, \quad m_1 \in S\}$. There are some well-known facts about the functions ν and δ_{FR} for an arbitrary semigroup S (see [3]). The most important one for our purposes is that $\delta_{FR}(m) \geq m + 1 - 2g$ for all $m \in S$ with $m \geq c$, and that equality holds if moreover $m \geq 2c - 1$.

The classical Feng-Rao distance corresponds to $r = 1$ in the following definition: For $m_1, \cdots, m_r \in S$, let $\mathrm{D}(m_1, \cdots, m_r) := \mathrm{D}(m_1) \cup \cdots \cup \mathrm{D}(m_r)$ and $\nu(m_1, \cdots, m_r) := \sharp\mathrm{D}(m_1, \cdots, m_r)$. Then for any integer $r \geq 1$, the *r-th Feng-Rao distance* (order bound) of S is defined by

$$\delta^r_{FR}(m) := \min\{\nu(m_1, \ldots, m_r) \mid m \leq m_1 < \ldots < m_r, \ \ m_i \in S\}.$$

Very few results are known for the numbers δ^r_{FR}, and their computation is very hard from both a theoretical and computational point of view. The main result we need describes the asymptotical behaviour for $m >> 0$, and states that there exists a constant $E_r = \mathrm{E}(S, r)$, depending on r and S, such that

$$\delta^r_{FR}(m) = m + 1 - 2g + E_r$$

for $m \geq 2c-1$. This constant is called the *r-th Feng-Rao number* of the semigroup S. Moreover, it is also true that $\delta^r_{FR}(m) \geq m+1-2g+\mathrm{E}(S,r)$ for $m \geq c$ (see [2]). For the trivial semigroup with $g=0$, it is easy to check that $\mathrm{E}(S,r)=r-1$, and for general $g \geq 1$ and $r \geq 2$ we have the bounds

$$r \leq \mathrm{E}(S,r) \leq \rho_r$$

and $\mathrm{E}(S,r)=\rho_r=r+g-1$ if furthermore $r \geq c$.

These concepts naturally arise from the theory of Algebraic Geometry codes for the case of Weierstrass semigroups, but here they are studied in general for any given numerical semigroup (see [3] for the details). The purpose of this paper is to compute the Feng-Rao numbers, and thus the Goppa-like bounds for the generalized Feng-Rao distances, in particular interesting cases of numerical semigroups.

2 Summary of results

The computation of the Feng-Rao numbers is a very hard task, even for very simple examples. Only the second Feng-Rao number ($r=2$) is computed in the literature. In fact, the only known general formula is $\mathrm{E}(S,2)=\rho_2$ when $\langle a,b\rangle$. In this paper we generalize this result to some particular cases of embedding dimension two numerical semigroups, namely

$$\mathrm{E}(S,r)=\rho_r$$

for hyper-elliptic semigroups $\langle 2, 2g+1\rangle$ and for Hermitian-like semigroups $\langle a, a+1\rangle$. This second case generalizes the Weierstrass semigroup of the so-called Hermitian curve, providing nice examples of error-correcting codes. Many examples have been tested with the aid of the `numericalsgps` GAP package [1].

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Moving rectangles *

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Consider a set of n pairwise disjoint axis-parallel rectangles in the plane. We call this set the source rectangles $\mathcal{S}$. The aim is to move the source rectangles to a set of (pairwise disjoint) target rectangles $\mathcal{T}$. A move consists in a horizontal or vertical translation of one rectangle, such that it does not collide with any other rectangle during the move. We say that two rectangles collide if their intersection is non-empty. We study how many moves are needed to transform $\mathcal{S}$ into $\mathcal{T}$. This problem can be seen in the context of motion planning.

In a related work, Abellanas et al. [1] studied the number of moves needed to translate coins. There a move consists of translating a coin along a fixed (not necessarily axis-parallel) direction. Another model for reconfiguration of disks, studied by Bereg, Dumitrescu and Pach [4], considers disk slides, where the center of a disk moves along an arbitrary continuous curve. Bereg and Dumitrescu [3] investigated the number of moves needed to reconfigure disks in the lifting model, where a disk is lifted from the plane and placed back in the plane at another location. Recent results on translating and sliding coins, and convex bodies in general, are due to Dumitrescu and Jiang [6]. They also show that in the translation model (not restricted to axis-parallel directions) of unlabeled axis-parallel unit squares $2n-1$ moves are always sufficient and $\lfloor \frac{3n}{2} \rfloor$ moves are sometimes necessary to transform $\mathcal{S}$ into $\mathcal{T}$. Related works on motion planning are also reconfigurations of modular metamorphic systems in a rectangular model [7] and reconfigurations of modular cube-style robots [2].

We study reconfigurations of rectangles with axis-parallel movements for three settings: We consider labeled rectangles of different dimensions. Here each source rectangle has its uniquely assigned target rectangle. Then we study labeled rectangles of equal dimensions. And finally we consider unlabeled rectangles. There, all rectangles are translates of a given rectangle and we are free to assign the source rectangles to the target rectangles. The following table summarizes the obtained bounds. A main tool in our proofs are known results on separability of convex bodies [9,10].

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	necessary moves	sufficient moves
labeled rectangles of different dimensions	$3n-1$	$4n-\frac{n}{8\log_2 n}$
labeled rectangles of equal dimensions	$3n-1$	$\frac{31n}{8}$
unlabeled rectangles	$2n+1$	$\frac{17n}{6}$

Table 1. The number of moves that are sometimes necessary (left) and the number of moves that are always sufficient (right) to reconfigure a set of n rectangles.

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Minimal strong digraphs

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Abstract. We introduce adequate concepts of expansion of a digraph and we characterize the class of minimal strong digraphs whose expansion preserves the property of minimality. We prove that every minimal strong digraph of order $n \geq 2$ is the expansion of a minimal strong digraph of order $n-1$ and we give sequentially generative procedures for the constructive characterization of the classes of minimal strong digraphs and strong digraphs. Finally we describe algorithms to compute unlabeled minimal strong digraphs and isospectral classes of minimal strong digraphs.

Key words: (minimal) strong digraphs, isospectral strong digraphs.

1 Introduction

In this article, we focus on the study of strongly connected digraphs containing the least possible number of arcs (minimal strong digraphs), that is, strongly connected digraphs which cease to be so if any one of its arcs is suppressed.

We are previously interested in the following problem [6]: given a real polynomial $p(x)$ with degree n, find necessary and sufficient conditions for the existence of a nonnegative matrix A of order n with characteristic polynomial $p(x)$. The irreducible realizations of $p(x)$ are identified with strongly connected digraphs associated to A [2]. The class of strong digraphs can easily be reduced to the class of minimal strong digraphs, so we are interested in this one.

Many classes of connected graphs and digraphs have constructive characterizations. In particular, for (minimal) 2-connected graphs and (minimal) strong digraphs different procedures have been described to construct larger (di)graphs from smaller (di)graphs [4,5,1]. The common basic idea of these procedures consists of adding paths between qualified vertices.

Bhogadi [1] gives a characterization of Cunningham's decomposition trees for minimal strong digraphs under X-joint composition [3] and uses this characterization to catalogue all the Cunningham prime minimal strong digraphs through 11 vertices and to implement an algorithm, for the generation of all minimal strong digraphs

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through 12 vertices. This approach was abandoned due to the difficulty of recognizing prime minimal strong digraphs.

All of these procedures have been defined so that the conditions under which minimality is preserved are not characterized. This is not a difficulty when proving the possibility of obtaining any minimal strong digraph from another smaller one (Hedetneimi [5] gives a proof), but it is a difficulty when constructing efficient and sequential procedures and algorithms.

2 Results

We introduce two suitable concepts of expansion of a digraph for a sequential construction of minimal strong digraphs and characterize the class of minimal strong digraphs whose expansion preserves the property of minimality. We show how every minimal strong digraph of order $n \geq 2$ is the expansion of a minimal strong digraph of order $n-1$ and describe procedures to do this.

We propose a sequentially generative procedure for the constructive characterization of the class of minimal strong digraphs and we give a simple procedure to build the set of strong digraphs of order $n \geq 3$ starting from the set of minimal strong digraphs of order $n-1$.

We implement an algorithm to compute unlabeled minimal strong digraphs following the construction of the previous sections. Another algorithm allows the digraphs and the characteristic polynomials of the isospectral classes of the minimal strong digraphs to be obtained.

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On the evaluation of the Tutte polynomial at the points $(1, -1)$ and $(2, -1)$

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Abstract. C. Merino proved recently the following identity between evaluations of the Tutte polynomial of complete graphs: $t(K_{n+2}; 1, -1) = t(K_n; 2, -1)$. In this work we extend this result by giving a large class of graphs with this property, that is, graphs G such that there exist two vertices u and v with $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$. The class is described in terms of forbidden induced subgraphs and it contains in particular threshold graphs.

The Tutte polynomial is one of the most studied polynomial graph invariants. For a graph $G = (V, A)$, it is given by

$$t(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(G)-r(A)} (y-1)^{|A|-r(A)},$$

where $r(A)$ is the *rank* of A, defined as $|V| - \kappa(G[A])$, where $\kappa(G[A])$ is the number of connected components of the subgraph $G[A] = (V, A)$ induced by A.

We refer to [2] for details about the many combinatorial interpretations of the evaluations of the Tutte polynomial of a graph in different points of the plane and also along several algebraic curves. A pair of interpretations especially related to our work are that $t(G; 2, 0)$ is the number of acyclic orientations of G and that $t(G; 1, 0)$ is the number of acyclic orientations of G with a unique fixed source. With this in mind, it follows that $t(K_{n+1}; 1, 0) = t(K_n; 2, 0)$ (in fact, the same is true of any graph G with a universal vertex).

Merino [1] proved a similar identity, which is the starting point of our work:

$$t(K_{n+2}; 1, -1) = t(K_n; 2, -1).$$

We wonder whether there are other graphs with this property, that is, we search graphs G with a pair of vertices u, v such that $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$. Merino's proof used generating functions. It is not very difficult to adapt his proof to show the property for complete bipartite graphs and for graphs that are the sum of

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a clique and a coclique. (By a clique we mean a complete graph, and by a coclique a graph with no edges; the sum operation adds all edges between the two summands.) Observe that if we take a graph K_2 and replace each vertex by either a clique or a coclique, and add all edges among them, we obtain complete bipartite graphs, complete graphs, or graphs that are the sum of a clique and a coclique. These examples are particular cases of the following theorem, which is our main result. (The neighbourhood of a vertex v is denoted by $N(v)$, and P_k is a path of length k.)

Theorem 1. *Let $G = (V, E)$ be a simple graph and u and v distinct vertices of G such that u and v are adjacent. Let $A = N(u) - N(v) - \{v\}$,$B = N(v) - N(u) - \{u\}$ and $C = N(u) \cap N(v)$, and suppose that $V = A \cup B \cup C \cup \{u, v\}$.*

Then $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$ if the following conditions hold:

(i) $G[A]$ and $G[B]$ are cocliques, and $G[C \cup \{u, v\}]$ is a clique;

(ii) there is no induced P_3 with one endpoint in A and the other in B, nor the complement of such a path;

(iii) there is no induced pair of disjoint edges $2P_2$ with endpoints in $A \cup B$ or induced P_4 with both endpoints in A or both endpoints in B.

Furthermore, if G satisfies these conditions then so does any graph obtained from G by replacing a vertex of $A \cup B \cup \{u, v\}$ by a coclique or a vertex of $C \cup \{u, v\}$ by a clique.

Besides the examples given above, the case $B = \emptyset$ gives the class of threshold graphs, which are those graphs with no induced P_4, C_4 or $2P_2$.

As for the proof, it follows a generating function approach. The key point is that it is possible to find the generating function for the Tutte polynomials of the family of graphs obtained from an initial graph G by replacing vertices by cliques and cocliques. Then, the relationship between evaluations at $(1, -1)$ and $(2, -1)$ is expressed as a differential equation, from whose solutions the statement of the theorem is read.

Finally, let us remark that there are graphs for which $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$ for some u, v and that are not described by the theorem above (for instance, a cycle of length 6, with u, v any two vertices at distance 2).

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Graphs with equal domination and 2-domination numbers ⋆

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For a graph G a subset D of the vertex set of G is a *k-dominating set* if every vertex not in D has at least k neighbors in D. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$.

In 1985 Fink and Jacobson gave the following theorem.

Theorem 1. (Fink, Jacobson [1], 1985) *If $k \geq 2$ is an integer and G is a graph with $k \leq \Delta(G)$, then*

$$\gamma_k(G) \geq \gamma(G) + k - 2.$$

The inequality given above is sharp. However, the characterization of the graphs attaining equality is still an open problem. In this paper, we recompile results concerning the case $k = 2$, where γ_k can be equal to γ. In particular, we present the following results.

Theorem 2. (Hansberg, Volkmann [4], 2007) *If G is a graph with $\delta(G) \geq 2$ and $\gamma(G) = \beta(G)$, then $\gamma_2(G) = \gamma(G)$.*

Hereby, $\beta(G)$ denotes the *covering number* of G.

Theorem 3. (Hansberg, Volkmann [4], 2007) *Let G be a cactus graph. Then $\gamma_2(G) = \gamma(G)$ if and only if G is a C_4-cactus.*

Let $\mathcal{H}$ be the family of graphs such that $G \in \mathcal{H}$ if and only if either G arises from a cartesian product $K_p \times K_p$ of two complete graphs of order p for an integer $p \geq 3$ by inflating every vertex but the ones on a transversal to a clique of arbitrary order, or G is a claw-free graph with $\Delta(G) = n(G) - 2$ containing two non-adjacent vertices of maximum degree.

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Theorem 4. (Hansberg, Randerath, Volkmann [3]) *Let G be a connected claw-free graph. Then $\gamma(G) = \gamma_2(G)$ if and only if $G \in \mathcal{H}$.*

Theorem 5. (Hansberg, Randerath, Volkmann [3]) *Let G be a line graph. Then $\gamma_2(G) = \gamma(G)$ if and only if G is either the cartesian product $K_p \times K_p$ of two complete graphs of the same cardinality p or G is isomorphic to the graph J depicted bellow.*

Finally, we note that the graphs of the family $\mathcal{H}$, as also the cactus graphs of Theorem 3, contain many induced cycles of length 4. This accumulation of induced C_4's can be found in every graph fulfilling equality in Fink and Jacobson's bound, as was noted by Hansberg in [2].

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Algoritmos de Borrado para Tablas de Dispersión con Direccionamiento Abierto

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Resumen. Las tablas de dispersión (*hash*) nos permiten almacenar conjuntos de elementos y realizar búsquedas, inserciones y borrados con coste $\Theta(1)$ (independiente del número de elementos del conjunto), en promedio. Para ello se usa una tabla $t[0..M-1]$ y una función de *hash* $h : U \to [0..M-1]$, siendo U el conjunto "universo" al cual pertenecen los n elementos que queremos guardar en la tabla de dispersión. La idea es almacenar cada elemento x en $t[h(x)]$, si es posible. Existen dos grandes familias de esquemas de dispersión, de acuerdo con la manera en que se resuelven las *colisiones*, es decir, las situaciones en las que queremos guardar un elemento en una posición ya ocupada. En los esquemas de *direccionamiento abierto*, dado un elemento x se recorre una secuencia de posiciones $i_0 = h(x)$, i_1, i_2, ... hasta que encontramos a x en un cierto $t[i_k]$ (búsqueda con éxito) o encontramos un sitio $t[i_k]$ vacío (búsqueda sin éxito). En este último caso el elemento x puede guardarse en $t[i_k]$.

Además de algunos problemas inherentes al direccionamiento abierto (p.e. el apiñamiento o *clustering*), otro problema lo constituyen los borrados. No basta con buscar el elemento a borrar y dejar el sito $t[i_k]$ que lo contiene vacío. En este artículo revisamos un elegante algoritmo de borrado para tablas de hash con sondeo lineal (*linear probing*), y analizamos su eficiencia. Este algoritmo, aunque conocido desde hace muchos años, ha sido por desgracia frecuentemente olvidado; todos los libros de texto sobre estructuras de datos explican tablas de hash, la inmensa mayoría explican el sondeo lineal, y sin embargo muy pocos explican el algoritmo de borrado.

Adicionalmente, proponemos un método eficiente de borrado real que sirve para otros muchos esquemas de direccionamiento abierto, p.e., el sondeo cuadrático (*quadrating hashing*). Es el primero que se propone de estas características, pues excepto el algoritmo de borrado para sondeo lineal, en todos los restantes esquemas de direccionamiento abierto el único método de borrado existente era mediante marcaje de las posiciones afectadas como "borradas", con algunas pequeñas variaciones.

Palabras clave: Análisis de algoritmos, tablas de dispersión, hash, direccionamiento abierto, borrado.

On the number of solutions of regular systems in sets with positive density in abelian groups

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Abstract. We prove a density version for the Ramsey statement that, for fixed r and sufficiently large n, every coloring of an abelian group of order n with r colors has nontrivial monochromatic solutions of the system $Ax = 0$, where A is an integer matrix with row–sum zero. Our result shows that asymptotically the number of solutions is in fact as large as it can be expected.

Key words: Removal lemma, regularity lemma, abelian groups, hypergraphs.

1 Extended abstract

The celebrated Theorem by Szemerédi on the existence of k–term arithmetic progressions in every set of integers with positive density has seen significant developments in recent years.

A new combinatorial proof has been recently set up by the work of Gowers [1], Nagel, Rödl and Schacht [4] or Tao [7] by extending the original Triangle Removal Lemma of Rusza and Szemerédi [6] to the context of hypergraphs. The introduction of a Removal Lemma for Abelian groups by Green [2] also opened the way to algebraic versions of this combinatorial result.

We consider an analogous statement here in the context of general abelian groups. A $(k \times m)$ integer matrix A is said to satisfy the *strong column condition* if the sum of the columns is the zero vector. The definition is related to the characterization by Rado [5] of partition regular systems. A linear system with integer coefficients is *partition regular* if every finite coloring of the integers has a monochromatic solution of the system. If the matrix of the system has the strong column condition, then it contains trivial monochromatic solutions. Our main result states that such system contains nontrivial monochromatic solutions and moreover, the number of nontrivial

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solutions is as large as one may expect. We give below the precise statement in the context of finite abelian groups.

Theorem 1. *Let A be an integer $(k \times m)$ matrix with the strong column property. For every $\epsilon > 0$ there is $\delta = \delta(\epsilon, A) > 0$ with the following property.*

Let G be an abelian group of order n. For every subset $X \subset G$ of cardinality $|X| \geq \epsilon|G|$ there are, at least, δn^{m-k} solutions of the system $Ax = 0$ with $x_1, \ldots, x_m \in X$.

Note that Theorem 1 can be seen as a density version of the Ramsey statement which ensures that, for fixed r and sufficiently large n, every coloring of an abelian group of order n with r colors has nontrivial monochromatic solutions of the system $Ax = 0$. By standard methods one can translate the above statement to torsion free Abelian groups.

The proof of Theorem 1 is a consequence of the following general result which has been recently proved by the authors. Recall that the k–determinantal of an integer matrix A, denoted by $d_k(A)$, is the greatest common divisor of all the determinants of square submatrices of A of order k.

Theorem 2 ([3]). *Given an integer $(k \times m)$ matrix A and $\epsilon > 0$ there is a $\delta = \delta(\epsilon, A)$ such that the following holds.*

For every Abelian group G of order n coprime with $d_k(A)$ and every family of subsets $X_1, \ldots, X_m$ of G, if the homogeneous linear system $Ax = 0$ has at most δn^{m-k} solutions with $x_1 \in X_1, \ldots, x_m \in X_m$ then there are sets $X_1' \subset X_1, \ldots, X_m' \subset X_m$ with $\max_i |X_i'| \leq \epsilon n$ such that there are no solutions to the system with $x_1 \in X_1 \setminus X_1', \ldots x_m \in X_m \setminus X_m'$.

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On the ASD conjecture ⋆

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Abstract. Let G be a graph of size $\binom{n+1}{2}$ for some integer $n \geq 1$. G is said to have an ascending subgraph decomposition (ASD) if can be decomposed into n subgraphs $H_1, \ldots, H_n$ such that H_i has i edges and is isomorphic to a subgraph of H_{i+1}, $i = 1, \ldots, n-1$. In this work we deal with ascending subgraph decompositions of bipartite graphs.

In order to do so, we consider ascending subgraph decompositions in which each factor is a forest of stars. We show that every bipartite graph G with $\binom{n+1}{2}$ edges such that the degree sequence $d_1 \geq \cdots \geq d_k$ of one of the partite sets satisfies $d_1 \geq (k-1)(n-k+1)$, and $d_i \geq n-i+2$ for $2 \leq i < k$, admits an ASD with star forests. We also give a necessary condition on the degree sequence of G to have an ascending subgraph decomposition into star forests that is not far from the above sufficient one. Our results are based on the existence of certain matrices that we call *ascending* and the construction of edge-colorings of some bipartite graphs with parallel edges.

Key words: Graph decomposition, edge-coloring, ascending matrix, sumset partition problem.

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Vértices centrales en los grafos radiales de Moore cúbicos

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Resumen. En el contexto del llamado *problema grado/diámetro*, se tiene una cota natural para el número de vértices de un grafo, fijado su grado máximo d y su diámetro k. Los grafos cuyo orden alcanza esta cota, $M_{d,k} = 1+d+d(d-1)+\cdots+d(d-1)^{k-1}$, conocida como *cota de Moore*, reciben el nombre de *grafos de Moore*. Estos grafos, que también son extremales para el *problema de las jaulas*, han sido estudiados en profundidad. En particular, el problema de su existencia está completamente resuelto excepto en un único caso (diámetro $k = 2$ y grado $d = 57$). Como, en general, hay muy pocos grafos de Moore, muchas de las lineas de investigación relacionadas con estos grafos se han dirigido a determinar de qué manera podemos aproximarnos a estos grafos. En este sentido, los estudios sobre los llamados grafos con *defecto* pequeño centran gran parte de estos esfuerzos (siendo el defecto la diferencia entre la inacalzable cota de Moore y el orden del grafo). Una aproximación distinta la constituyen los denominados *grafos radiales de Moore*. Estos grafos aparecen cuando se fija como número de vértices la cota de Moore, dada ahora en función del radio k (en lugar del diámetro), pero se relaja la condición fuerte de que todo vértice tenga la mínima excentricidad (k). En este sentido, diremos que un grafo regular de grado d, radio k, diámetro $k+1$ y con $M_{d,k}$ vértices es un grafo radial de Moore. Esta relajación en el diámetro provoca que los vértices puedan tener excentricidad k (vértices centrales) o bien $k+1$. En este trabajo mostraremos determinadas configuraciones imposibles entre vértices centrales y no centrales en un grafo radial de Moore, lo cual nos permitirá deducir una cota superior para el número de vértices centrales en el caso cúbico.

Problemas abiertos sobre etiquetamientos super edge-magic y temas relacionados

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Palabras clave: super edge-magic, deficiencia, magic model, productos mágicos

1 Resumen

Un etiquetamiento de un grafo es una asignación de elementos de cierto conjunto (normalmente el de los enteros) a los vértices, a las aristas, o a los vértices y a las aristas del grafo, que cumple ciertas propiedades.

Rosa en el año 1967, introduce los etiquetamientos graciosos como una manera de atacar la conjetura de Ringel, que afirma que cualquier árbol T de orden p descompone al grafo completo K_{2p+1} en $2p+1$ copias de T. Hoy en día esta conjetura sigue abierta, y sólo casos particulares de la misma han conseguido resolverse.

Los etiquetamientos graciosos son el origen de otros tipos de etiquetamientos, destacamos, por el número de artículos publicados, los etiquetamientos armónicos introducidos por Graham y Sloan en 1980.

Enomoto, Lladó, Nakamigawa y Ringel introducen en 1998 los etiquetamientos super edge-magic. Aunque en un artículo anterior, Acharya y Hedge habían definido un etiquetamiento equivalente, usando progresiones aritméticas. En esta charla nos concentraremos básicamente en etiquetamientos super edge-magic que guardan fuerte relación con los graciosos y los armónicos. A continuación y con el objetivo de hacer este resumen razonablemente autocontenido, definimos los grafos y etiquetamientos graciosos, armónicos y super edge-magic.

Sea $G = (V, E)$ un grafo de orden p y medida q. Un etiquetamiento *gracioso* de G es una función inyectiva f del conjunto de vértices en el conjunto $\{0, 1, \ldots, q\}$ tal que, cuando se asigna a cada arista xy la etiqueta $|f(x) - f(y)|$, las etiquetas obtenidas en las aristas son distintas. Los grafos que admiten este tipo de etiquetamientos se llaman graciosos. El grafo G se llama *armónico* si existe una función inyectiva f del

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conjunto de vértices en el grupo de enteros módulo q tal que cuando se asigna a cada arista xy la etiqueta $f(x)+f(y)$ (mod q), las etiquetas resultantes son distintas. Cuando G es un árbol, exactamente una etiqueta puede usarse en dos vértices. El grafo G es *super edge-magic* si existe una función biyectiva $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ tal que (1) $f(V) = \{i\}_{i=1}^{p}$ y (2) $f(x) + f(xy) + f(y) = k$ para toda arista xy.

El objetivo de este trabajo es presentar algunos problemas abiertos vinculados a los etiquetamientos super edge-magic.

En primer lugar consideraremos el problema de la máxima densidad de un grafo super edge-magic, esto es, dado un grafo $G = (V, E)$ con orden p y medida q, ¿podemos acotar q en función de p?. Enomoto, Lladó, Nakamigawa y Ringel, demuestran la desigualdad $q \leq 2p - 3$ y Figueroa et al. que los grafos super edge-magic que alcanzan la igualdad han de contener triángulos. Todo indica que aumentando la cintura del grafo, la medida de los grafos super edge-magic disminuye. En la Sección 2, abordamos esta cuestión y presentamos algunas posibles direcciones de investigación.

Sea $G = (V, E)$ un grafo y sea $M(G)$ un conjunto de enteros positivos, incluido el cero. Un entero n pertenece al conjunto $M(G)$ si $G \cup nK_1$ es super edge-magic. Se define la *deficiencia super edge-magic* de G como el mínimo de $M(G)$, si el conjunto es no vacío, e infinito en caso contrario. En la Sección 3, presentamos algunos de los resultados más significativos sobre la deficiencia super edge-magic e introducimos algunas preguntas que sugieren diferentes líneas de estudio.

Recientemente se ha generalizado el concepto de etiquetamiento super edge-magic al concepto de etiquetamiento super edge-magic respecto a un modelo. Esta generalización demuestra que el concepto de super egde-magic tiene un fuerte vínculo con la coloración propia de aristas de un grafo. En la Sección 4, presentaremos algunas de las preguntas que surgen de esta nueva vía.

En la Sección 5, consideramos la relación entre el clásico problema de las reinas y los etiquetamiento a los que dan nombre.

En la Sección 6, estudiamos los etiquetamientos producto mágicos y producto antimágicos, que hasta la fecha han sido analizados en gran medida usando métodos probabilísticos.

Polinomios característicos de digrafos pesados y desigualdades de Newton

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Palabras clave: Digrafos pesados, matrices no negativas, desigualdades de Newton, problemas espectrales inversos no negativos.

1 Resumen

Holtz [1] observa que si $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ es el espectro de una matriz no negativa de radio espectral ρ, entonces $\rho - \sigma = \{\rho - \lambda_1, \ldots, \rho - \lambda_n\}$ es un espectro Newton, es decir, $\Delta_j(\rho - \sigma) \geq 0,\ j = 1, \ldots, n-1$. Las desigualdades de Newton Δ_j se definen para el espectro σ de una matriz real como $\Delta_j(\sigma) = c_j(\sigma)^2 - c_{j-1}(\sigma)c_{j+1}(\sigma) \geq 0$, para $j = 1, \ldots, n-1$, donde $c_j(\sigma)$ es la j-ésima función simétrica normalizada de σ. Esta condición necesaria para el problema espectral inverso no negativo pone de manifiesto el interés de caracterizar los espectros Newton y de relacionar los espectros Newton con los problemas espectrales inversos no negativos.

El problema espectral inverso no negativo consiste en, dada una familia $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ de n números complejos, encontrar condiciones necesarias y suficientes para la existencia de una matriz no negativa A de orden n con espectro σ. Una matriz no negativa puede verse como la matriz de adyacencia de un digrafo pesado en los arcos, y como el espectro determina el polinomio característico, nosotros proponemos reconstruir digrafos pesados a partir de su polinomio característico. Así otro planteamiento de este problema es: dados n números reales $k_1, k_2, \ldots, k_n$, encontrar condiciones necesarias y suficientes para la existencia de un digrafo pesado de orden n con polinomio característico $x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_n$. Este problema está abierto para $n \geq 5$. Notemos que las identidades de Newton relacionan los coeficientes $k_j = k_j(\sigma)$ con los momentos $s_j = s_j(\sigma) = \sum_{i=1}^{n} \lambda_i^j$ del espectro σ y que el teorema de los coeficientes permite obtener los coeficientes del polinomio característico a partir de la estructura cíclica del digrafo.

El problema de las desigualdades de Newton trata de caracterizar las familias de n números complejos $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ autoconjugadas que satisfacen las desigualdades de Newton. Es sabido que si σ es real o es el espectro de una M-matriz o de

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una M-matriz inversa, entonces verifica las desigualdades de Newton. Este problema está abierto para $n \geq 4$.

Nuestros principales resultados son:

Teorema. *Sean $a, b, c \in \mathbb{R}$ con $c > 0$. El espectro $a, b \pm ic$ es Newton si, y solo si, $|a - b| \geq \sqrt{3}c$ y $|c^2 + b(b - a)| \geq \sqrt{3}c|a|$.* (Ver [2])

Teorema. 1. *Sean $a, b, c \in \mathbb{R}$ con $c > 0$. El espectro $a, b, \pm ic$ es Newton si, y solo si, $\{a, b\} = \{-c, c\}$.* (Ver [2])
2. *Sean $a, b, c, d \in \mathbb{R}$ con $b, d > 0$. El espectro $a \pm ib, c \pm id$ no es Newton si $a = c$ ó si $ac = 0$.* (Ver [2])
3. *Sean $a, b, c \in \mathbb{R}$ con $c > 0$. El espectro $a, -a - 2b, b \pm ic$ es Newton si, y solo si, $a^2 + 2ab + 3b^2 \geq c^2$ y $3b^2(b^2 + c^2)^2 + a(a + 2b)[2c^2(b^2 + c^2) + a(a + 2b)(b^2 - 2c^2)] \geq 0$.*

Teorema. *Sea $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ una familia autoconjugada de números complejos con radio espectral ρ. Las siguientes condiciones son equivalentes:*

1) La condición de Johnson-Loewy-London para $j = 1$ y $m = 2$, es decir, $ns_2(\sigma) - s_1(\sigma)^2 \geq 0$.

2) La condición sobre el segundo coeficiente, es decir, $k_1(\sigma)^2 - \frac{2n}{n-1}k_2(\sigma) \geq 0$.

3) La primera de las condiciones de Holtz, es decir, $\Delta_1(\rho - \sigma) \geq 0$.

Utilizando los resultados anteriores y otros bien conocidos en la literatura acerca de problemas espectrales inversos no negativos hemos establecido relaciones de dependencia e independencia entre ambos problemas en los casos $n \leq 3$, $n = 4$ con traza cero y $n = 5$ con traza cero.

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Avances sobre los números de Schur estrictos y los números de Rado estrictos

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Resumen. En este trabajo se obtienen cotas inferiores de los *números de Schur estrictos* y se calculan los *números de Rado estrictos* para dos y tres subconjuntos *estrictamente libres de suma*.

Palabras clave: Número de Schur, número de Rado y número de Ramsey

1 Introducción

En 1916, Schur [19] planteó un problema relacionado con el Teorema de Fermat, buscando particiones del conjunto $\{1, 2, ..., N\}$ en n partes de tal forma que ninguna de las partes contuviese dos elementos y su correspondiente suma. Sin embargo obtener los valores exactos de los *números de Schur* es un problema sumamente difícil de resolver. En 1961, Baumert [2] encontró el último valor exacto conocido hasta el momento para $n = 4$, el problema aún sigue abierto para $n \geq 5$. Desde entonces se ha ido avanzando obteniéndose cotas superiores e inferiores de estos números. Existen estudios realizados por diferentes autores tales como Abbott y Hanson (1972) [1], Whitehead (1973) [21], Exoo (1994) [7], Radziszowki (1999) [15], Fredricksen y Sweet (2000) [8], que determinan estas cotas.

Es de destacar que los *números de Schur* acotan superiormente a los *números de Ramsey*. Esta relación obtenida en el año 2000 por Robertson [16] es de gran relevancia, puesto que el avance en los *números de Schur* puede dar lugar a nuevos avances en los *números de Ramsey*, ó bien con los progresos que vayan surgiendo de los *números de Ramsey* se pueden mejorar las cotas superiores de los *números de Schur*.

En 1933, Rado [13], [14] que se doctoró bajo la dirección de Schur, consideró el problema de determinar si un sistema de ecuaciones diofánticas admite solución monocromática para cada *n-coloración* de números naturales, generalizando así el problema de los *números de Schur*. Después de 76 años de los primeros resultados de Rado, se han obtenido muy pocos progresos.

Burr y Loo en [5] [6] investigaron algunas familias particulares de ecuaciones con *2-coloración* de enteros positivos. En 1993, Schaal [17] generaliza la definición del *número de Rado* para k sumandos. Además, en 1995 [18], obtuvo el *número de Rado* para particiones del conjunto $\{1, 2, , ..., N\}$, formada por tres conjuntos *libres de suma*.

Desde el 2001 hasta el 2008, diferentes autores, Kosek y Schaal [12], Guo y Sun [10] obtienen cotas inferiores y superiores y valores exactos de algunos *números de Rado* ó variaciones del problema.

En este trabajo se presentan resultados sobre los *números de Schur estrictos* y los *números de Rado estrictos*, una variante de los problemas anteriormente mencionados, donde la condición sobre cada subconjunto de la partición del conjunto $\{1, 2, ..., N\}$, es no contener elementos $x_1, x_2, ..., x_k$ junto con su suma $x_1+x_2+..+x_k+c$. Estudiamos los casos para $c = 0$ y $c > 0$ y los x_i todos distintos entre sí.

Del caso $c = 0$ y los x_i todos distintos entre sí, que hemos denominado *números de Schur estrictos*, los únicos resultados conocidos hasta el momento se deben a Sierpinski [20] en 1964, a Irving [11] en 1973 y a Bornsztein [4] en 2002 que obtiene una cota superior.

En nuestro trabajo, abordamos la difícil tarea de calcular valores exactos de los *números de Schur estrictos*, cotas inferiores y superiores generales para ellos.

Del caso $c > 0$ y los x_i todos distintos entre sí, que hemos denominado *números de Rado estrictos*, obtenemos soluciones monocromáticas de la ecuación $x_1+x_2+c = x_3$ para una *2-coloración* y una *3-coloración* del conjunto $\{1, 2, ..., N\}$.

2 Números de Schur

Si designamos por $\{1, 2, ..., N\}$ a un conjunto de enteros, el *número de Schur* denotado por $S(n, 2)$, se define como el mayor entero $N + 1$ tal que el conjunto $\{1, 2, ..., N\}$ pueda particionarse en n subconjuntos $\{A_1, A_2, ..., A_n\}$ *libres de suma*, es decir:

$$\forall x_1, x_2 \in A_i \Rightarrow x_1 + x_2 \notin A_i, i = 1, 2, ..n$$

donde x_1 y x_2 **no necesariamente distintos**.

En 1916, Schur [19] determina la siguiente acotación:

$$1/2(3^n + 1) \leq S(n, 2) \leq [n!e]$$

donde $[x]$ denota el mayor entero que no exceda de x.

En 1972, Abbott y Hanson [1] mejoraron la cota inferior

$$S(n, 2) - 1 \geq c89^{\frac{k}{2}}$$

siendo c una constante positiva. La cota superior fue mejorada en 1973 por Whitehead [21] usando la Teoría de Ramsey.

$$S(n, 2) \leq [n!(e - \frac{1}{24})]$$

En 1994, Exoo [7] obtuvo la siguiente acotación para el *número de Schur* $S(5, 2)$:

$$161 \leq S(5, 2) \leq 322$$

La cota superior fue mejorada en 1999 por Radziszowki [15]

$$S(5, 2) \leq 316$$

Para particiones de seis y siete conjuntos *libres de suma* en el 2000 Fredricksen y Sweet [8] construyeron particiones en las que obtuvieron las cotas siguientes:

$$S(6,2) \leq 537$$
$$S(7,2) \leq 1681$$

Una extensión natural del concepto de conjunto *libre de suma* se obtiene en la siguiente definición, para k sumandos:

Un conjunto A de enteros se dice *libres de k-sumas* si

$$\forall x_1, x_2, ..., x_k \in A \Rightarrow x_1 + x_2 + .. + x_k \notin A$$

donde los x_i **no necesariamente son distintos**.

La definición de *número de Schur* se generaliza para n subconjuntos con k sumandos *libres de suma*, y se denota por $S(n,k)$.

En 1966, Zánm [22] obtiene una cota inferior que generaliza la dada por Schur:

$$S(n,k) \geq (\frac{k-1}{k})((k+1)^n - 1) + 1$$

También obtiene una cota superior utilizando la Teoría de Ramsey

$$S(n,k) \leq R(n,k+1) - 1$$

Por tanto usando un conocido resultado de Greenwood y Gleason [9] de 1955, se obtiene

$$S(n,k) \leq \frac{(nk)!}{(k!)^n} - 1$$

Znám [23] probó además que es óptima para $n = 2$.

Irving [11] en 1973 mejora la cota superior dada por Znám obteniendo que:

$$S(n,k) \leq [n!(k-1)^n exp(\frac{1}{(k-1)})]$$

En 1982, Beutelspacher y Brestovansky [3] demuestran que para dos conjuntos *k-libres de suma* se verifica la siguiente igualdad:

$$S(2,k) = k^2 + k - 1,\ k \geq 2$$

En la siguiente tabla representamos los números de Schur $S(n,k)$, para valores determinados de n y k, conocidos hasta el momento.

$$S(n,k)$$

k	**2**	**3**	**4**	**5**	**k**
n=2	5	11	19	29	$k^2 + k - 1$
n=3	14				$\geq k^3 + 2k^2 - 2$
n=4	45				

3 Número de Schur estricto

En nuestro trabajo consideramos una variante del problemmaa anterior. Llamamos *números de Schur estrictos* y los denotaremos por $HS(n,k)$, al mayor entero $N+1$ tal que el conjunto $\{1,2,...,N\}$ puede particionarse en n subconjuntos $\{A_1,A_2,...,A_n\}$ *estrictamente libres de k-sumas*, es decir :

$$\forall x_1,x_2,...,x_k \in A_i \Rightarrow x_1+x_2+..+x_k \notin A_i$$

con $i=1,2,...,n$ y donde los x_i son todos **distintos**.

En 1973 Irving obtuvo una cota superior de los $HS(n,k)$:

$$HS(n,k) \leq [\frac{1}{2}n!(k-1)^n(kn+1)exp(\frac{1}{k-1})+\frac{k}{k-1}]$$

Nuestras aportaciones consisten en calcular valores exactos de los *números de Schur estrictos* y cotas inferiores generales para ellos.

Obtenemos la cota inferior de los *números de Schur estrictos* para n conjuntos *estrictamente libres de k-sumas*. Esta cota depende de n y de k .

Teorema 1.

$$HS(n,k) \geq \frac{1}{2}(k+3)k^n - k$$

siendo $n \geq 2$ *y* $k \geq 2$

Mejoramos esta cota inferior ampliando los bloques utilizados en la demostración del Teorema 1. De esta forma obtenemos el siguiente resultado, que determina una relación entre los *números de Schur estrictos* para r y $r+1$ conjuntos *estrictamente libres de k-sumas*

Lema 1.

$$HS(r+1,k) \geq kHS(r,k)+(k-1)p(k)$$

siendo $p(k)=\frac{1}{2}(k^2+5k-2)$
y $r \geq 1$

La cota de los los *números de Schur estrictos* para n conjuntos *estrictamente libres de k-sumas*, que mejora la obtenida previamente se obtiene en el resultado siguiente.

Lema 2.

$$HS(n,k) \geq k^{n-1}(T_k+k-1)+(k^{n-2}-1)p(k)$$

siendo $p(k)=\frac{1}{2}(k^2+5k-2)$
y $r \geq 1$

Con la aplicación del lema anterior se obtiene la mejora de la cota del Teorema 1

Teorema 2.

$$HS(n,k) \geq q(k)k^{n-2} - p(k)$$

siendo $q(k) = \frac{1}{2}k^3 + 2k^2 + \frac{3}{2}k - 1$

y $p(k) = \frac{1}{2}(k^2 + 5k - 2)$

Esta expresión determina las cotas inferiores de los *números de Schur estrictos* para n conjuntos con k sumandos *estrictamente libres de suma* para $n \geq 2$ y $k \geq 2$.

Obtenemos con el siguiente resultado la acotación de los *números de Schur estrictos* para $n = 4$ conjuntos $\{A_1, A_2, A_3, A_4\}$, *estrictamente libres de k-sumas*:

Lema 3.

$$HS(4,k) \geq [k^3 + 4k^2 + 6k + 3]T_k - 2k^3 - 7k^2 - 7k + 3$$

Generalizando la construcción realizada en el Lema 3, obtenemos la expresión que relaciona los *números de Schur estrictos* para r y $r+1$ conjuntos *estrictamente libres de k-sumas.*

Lema 4.

$$HS(r+1,k) \geq kHS(r,k) + p(k)$$

siendo $p(k) = (k-1)(\frac{1}{2}k^3 + \frac{5}{2}k^2 + \frac{7}{2}k - 3)$
y $r \geq 3$

Utilizando la relación del Lema 4, obtenemos la generalización para n conjuntos *estrictamente libres de k-sumas* , mejorando el resultado del Teorema 2

Teorema 3.

$$HS(n,k) \geq q(k)k^{n-3} - p(k)$$

siendo $p(k) = \frac{1}{2}k^3 + \frac{5}{2}k^2 + \frac{7}{2}k - 3$

y $q(k) = \frac{1}{2}k^4 + \frac{5}{2}k^3 + \frac{7}{2}k^2 - 2$

para $n \geq 3$ *y* $k \geq 2$

4 Número de Rado

En 1933 Rado [13][14] consideró el problema de determinar si un sistema de ecuaciones lineales admite solución monocromática para cada *n-coloración* de nú meros naturales.

En términos de coloraciones, el *número de Rado* $R(2,c)$ es el mínimo entero tal que para cada *2-coloración* del conjunto $\{1,2,,...,N\}$:

$$\Delta : \{1,2,...,N\} \longrightarrow \{0,1\}$$

existe una solución monocromática de la ecuación

$$x_1 + x_2 + c = x_3$$

donde x_1 y x_2 no son necesariamente distintos.
Bur y Loo [6] demostraron que si $c \geq 0$ el *número de Rado*

$$R(2,c) = 4c + 5$$

Schaal [17] en 1993 generaliza la definición del *número de Rado* para k sumandos.

El *número de Rado* $R(k,c)$ para $k \geq 2$ y $c \geq 0$ es el mínimo entero tal que para cada *2-coloración* del conjunto $\{1,2,,...,N\}$:

$$\Delta : \{1,2,...,N\} \longrightarrow \{0,1\}$$

existe una solución monocromática a

$$x_1 + x_2 + ...x_k + c = x_{k+1}$$

donde $x_1, x_2, ..x_k$ no son necesariamente distintos.
En el caso de que no exista se define $R(k,c) = \infty$

Schaal [17] obtiene el siguiente resultado :

Teorema 4. *[17]*

Si $k \geq 0$ *y* $c \geq 0$ *entonces el número de Rado es:*

$$R(k,c) = \begin{cases} \infty & k,c \text{ impares} \\ (k+1)^2 + (c-1)(k+2) & \text{otro caso} \end{cases}$$

Los resultados de Bur y Loo [6] se obtienen como un corolario del Teorema 4 para $k = 2$.

En 1995, Schaal [18] obtuvo el *número de Rado* para particiones del conjunto $\{1,2,,...,N\}$, formada por tres conjuntos *libres de suma en el sentido de Rado.*

El *número de Rado* $R(3,c)$ es el mínimo entero tal que para cada *3-coloración* del conjunto $\{1,2,,...,N\}$

$$\Delta : \{1,2,...,N\} \longrightarrow \{0,1,2\}$$

existe una solución monocromática a la ecuación:

$$x_1 + x_2 + c = x_3$$

donde x_1 y x_2 no son necesariamente distintos.

Schaal demuestra $R(3,c) = 13c + 14$ para $c \geq 0$

En 2001 Kosek y Schaal [12] obtienen cotas inferiores y superiores de los *números de Rado* , para algunos valores determinados de $c < 0$.

5 Número de Rado estricto

Definimos el *número de Rado estricto* $SR(2,c)$ como el mayor entero positivo tal que el conjunto $\{1,2,,...,N\}$ pueda particionarse en dos subconjuntos *estrictamente libres de suma en el sentido de Rado.*

El conjunto A es *estrictamente libre de suma en el sentido de Rado* si:

$$\forall x_1, x_2 \in A \Rightarrow x_1 + x_2 + c \notin A$$

siendo x_1 y x_2 distintos.

Como hemos indicado anteriormente, en 1995, Schaal [18] obtuvo $R(3,c) = 13c + 14$ para $c \geq 0$ y en 2001, Kosek y Schaal [12] obtienen cotas inferiores y superiores para algunos valores determinados de $c < 0$. Desde entonces no se han producido nuevos avances en el cálculo de valores exactos ó cotas del problema.

En este trabajo nos planteamos la variante de los *números de Rado estrictos* definido previamente, cuya única diferencia con el problema original es exigir la condición de que x_1 y x_2 *sean distintos*, encontrando el valor exacto de $SR(2,c) = 4c + 8$ y $SR(3,c) = 13c + 22$.

Lema 5.

$$SR(2,c) \geq 4c + 8$$

con $c > 0$

Lema 6.

$$SR(2,c) \leq 4c + 8$$

con $c \geq 0$

Los dos conjuntos *estrictamente libres de suma en el sentido de Rado* son:

$$A_1 = \{1, 2, ..c + 2, 3c + 7, ..., 4c + 7\}$$

$$A_2 = \{c + 3, c + 4, ..., 3c + 6\}$$

Con los resultados de los Lemas 5 y 6 obtenemos:

Teorema 5.

$$SR(2, c) = 4c + 8$$

Continuando con este estudio obtendremos el *número de Rado estricto* $SR(3, c)$, es decir el mayor entero positivo tal que el conjunto $\{1, 2, ..., N\}$ puede particionarse en tres subconjuntos *estrictamente libres de suma en el sentido de Rado.*

En términos de coloraciones el *número de Rado estricto*, $SR(3, c)$ es el mínimo entero tal que para cada *3-coloración* del conjunto $\{1, 2, , ..., N\}$

$$\Delta : \{1, 2, ..., N\} \longrightarrow \{0, 1, 2\}$$

existe una solución monocromática a la ecuación:

$$x_1 + x_2 + c = x_3,$$

siendo $x_1 \neq x_2$ y $c > 0$.

Teorema 6.

$$SR(3, c) = 13c + 22$$

para $c > 0$

Los tres conjuntos *estrictamente libres de suma* en el sentido de Rado son

$A_1 = \{1, ..c + 2, 3c + 7, .., 4c + 7, 9c + 17, .,10c + 17, 12c + 21, ..., 13c + 21\}$

$A_2 = \{c + 3, ..., 3c + 6, 10c + 18, ..., 12c + 20\}$

$A_3 = \{4c + 8, 4c + 9, ..., 9c + 16\}$

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Programación lineal modular y bases de Graver: cálculo de soportes minimales de códigos lineales

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Este trabajo está motivado por la relación del problema de la descodificación completa con ciertos problemas de programación entera y su objetivo es describir algebraicamente el conjunto de palabras de soporte mínimo de un código definido sobre $\mathbb{Z}_q$ y su cálculo (o equivalentemente, circuitos mínimos en un matroide $\mathbb{Z}_q$-representable). El interés del conjunto de palabras de soporte mímimo se origina con los algoritmos de descodificación por gradiente [1,6] recientemente unificados mediante el concepto de representación de Gröbner [3]. Dicho conjunto también es de inters el el campo de los esquemas para compartir secretos basados en códigos correctores de errores ya que describen el conjunto de coaliciones minimales que acceden al mismo, véase el artículo seminal de J. L. Massey [5]. Desafortunadamente el cálculo de las palabras minimales o de mínimo soporte para un código lineal arbitrario es NP-completo incluso si se permite preprocesamiento pues es equivalente a la descodificación completa [2,4]. En este trabajo mostramos la relación algebraica entre ciertos ideales binomiales asociados a problemas de programación lineal entera modular y las palabras de soporte mínimo de ciertos códigos definidos por la matriz de restricciones lineales del problema.

Sea $q \geq 2$ un entero, a partir de ahora utilizaremos las siguientes aplicaciones de *cambio de característica*: $\blacktriangledown : \mathbb{Z}^s \to \mathbb{Z}_q^s$ y $\blacktriangle : \mathbb{Z}_q^s \to \mathbb{Z}^s$ donde s se determinará del contexto y las aplicaciones actúan por la derecha sobre vectores y matrices. La aplicación $\blacktriangledown$ corresponde a la reducción módulo q, y $\blacktriangle$ sustituye la clase de los elementos $0, 1, \ldots, q-1$ por el mismo símbolo considerado como entero. Ambas aplicaciones actúan coordenada a coordenada.Consideremos $A \in \mathbb{Z}_q^{m\times n}$ y los vectores $\mathbf{b} \in \mathbb{Z}_q^m$, $\mathbf{w} \in \mathbb{R}^n$, definimos el *problema de programación lineal modular*, $\mathrm{IP}_{A,\mathbf{w},q}(\mathbf{b})$ como el problema de encontrar el un vector $\mathbf{u} \in \mathbb{Z}_q^n$ que minimice $\mathbf{w} \cdot \blacktriangle\mathbf{u}$ y que cumpla $A\mathbf{u}^t \equiv b \bmod q$. Los dos principales resultados del trabajo son:

Proposición 1. *Sea $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subseteq \mathbb{Z}_q^n$ un subconjunto de $\mathbb{Z}_q$-generadores del espacio de filas de la matriz $A^\perp \in \mathbb{Z}_q^{(n-k)\times n}$ y consideremos dos vectores $\mathbf{a},\ \mathbf{b} \in \mathbb{Z}^n$. Las siguientes condiciones son equivalentes:*

1. $(\blacktriangle A) \cdot \mathbf{a} \equiv (\blacktriangle A) \cdot \mathbf{b} \bmod q$,
2. $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \in I(\blacktriangle A) := \{\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \mid (\blacktriangle A) \cdot \mathbf{a} \equiv (\blacktriangle A) \cdot \mathbf{b} \bmod q\}$.
3. $\exists \mathbf{t}_1, \mathbf{t}_2 \in \mathbb{K}[\mathbf{x}]$ *y* $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}^n$ *tales que*

$$\mathbf{x}^{\mathbf{a}+(q-1)\mathbf{b}}\mathbf{t}_1^q = \mathbf{t}_2^q \prod_{j=1}^{s} \mathbf{x}^{\lambda_i \blacktriangle \mathbf{w}_j}.$$

Proposición 2.

$$I(\blacktriangle A) = \left\langle \{\mathbf{x}^{\blacktriangle \mathbf{w}_1} - 1, \ldots, \mathbf{x}^{\blacktriangle \mathbf{w}_k} - 1\} \cup \{x_i^q - 1\}_{i=1}^n \right\rangle \subseteq \mathbb{K}[\mathbf{x}] \tag{1}$$

Consideremos un código lineal $\mathfrak{C} \subseteq \mathbb{Z}_q^n$. La aplicación principal de los resultados anteriores es

Corolario 1. *Las palabras minimales de* $\mathfrak{C}$ *se pueden calcular a partir del ideal generado por los binomios*

$$\{\mathbf{x}^{\blacktriangle \mathbf{w}_1}\mathbf{z}^{\blacktriangle \mathbf{w}_1(q-1)} - 1, \ldots, \mathbf{x}^{\blacktriangle \mathbf{w}_k}\mathbf{z}^{\blacktriangle \mathbf{w}_k(q-1)} - 1\} \cup \{x_i^q - 1\}_{i=1}^n \cup \{z_i^q - 1\}_{i=1}^n \tag{2}$$

donde $\mathbf{w}_i$, $i = 1, \ldots, k$ *son las filas de una matriz generadora del código* $\mathfrak{C}$.

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Etiquetamiento y enrutamiento de una familia de grafos planares *

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1 Resumen

En este trabajo presentamos un etiquetamiento y un algoritmo de enrutamiento óptimo para una familia de grafos planares, modulares y con apiñamiento nulo. Estas familias de grafos estan definidas por dos parámetros y su construcción es determinista con lo que podemos obtener expresiones analíticas de parámetros importantes que caracterizan las redes reales. Los grafos presentan propiedades similares a ciertas redes reales tecnológicas y biológicas con apiñamiento bajo, como algunas redes asociadas a circuitos electrónicos y las redes de proteinas.

Para estas redes es de gran importancia que existan modelos con un protocolo óptimo de enrutamiento para diseñar algoritmos de comunicación así como para entender los mecanismos que han formado sus estructuras.

Entre los trabajos más relevantes sobre redes complejas cabe destacar los de Watts y Strogatz sobre redes pequeño mundo y los de Baraba≪si y Albert sobre redes invariantes de escala. Estos trabajos han llevado a muchos investigadores a diseñar modelos para poder describir redes asociadas a sistemas complejos en la naturaleza y la sociedad como Internet, interacciones proteina-proteina, sistemas de transporte y redes sociales o económicas.

Éstos modelos pretenden explicar resultados observados experimentalmente. Numerosos estudios muestran la presencia de al menos tres características importantes: distancia media entre nodos y diámetro, pequeño, la distribución de enlaces por nodo sigue una ley de tipo potencial (redes invariantes de escala o *scale-free*); y recientemente se ha descubiero que a menudo las redes reales son autosimilares con una jerarquía que está relacionada con la modularidad de las redes modelos.

Muchos de los modelos propuestos inicialmente son estocásticos, aunque últimamente también se han propuesto modelos deterministas. Éstos últimos tienen la ventaja que permiten determinar de forma exacta el valor de algunos parámetros de la red. De entre los distintos modelos deterministas basados en métodos iterativos cabe destacar los pseudo-fractales en los que en cada iteración se añaden nuevos vértices de acuerdo con ciertas reglas. En otros trabajos en lugar de añadir vértices se añaden subgrafos completos. También hay trabajos en los que en cada iteracción se multiplica la estructura, o bien se añade una nueva estructura.

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Aquí hemos estudiado una familia de grafos que son planares, modulares, jerárquicos y autosimilares, con características *small-world* y con apiñamiento cero. Estas propiedades se corresponden con las observadas en importantes redes reales como por ejemplo algunas redes asociadas a circuitos eléctricos, Internet y algunos sistemas biológicos. Así pués, el enrutamiento estudiado en estas familias de grafos es una nueva herramienta para estudiar estos sistemas complejos y poder hallar nuevos algoritmos relacionados con procesos dinámicos.

Caminos de isogenias entre curvas elípticas sobre cuerpos finitos ⋆

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Palabras clave: criptografía, curvas elípticas, isogenias.

Resumen. Dadas dos curvas elípticas E_1 y E_2 sobre un cuerpo finito $\mathbb{F}_q$, una isogenia entre ambas es un morfismo que preserva la ley de grupo. Tate [6] prueba la existencia de una isogenia entre ellas si y sólo si las dos tienen el mismo cardinal $N = q + 1 - t$, $|t| \leq 2\sqrt{q}$. Sin embargo encontrar explícitamente una isogenia entre ambas curvas es un problema computacionalmente *duro*. Esta dificultad está en la base de propuestas de sistemas criptográficos de clave pública como [4]. Este criptosistema, basado en un camino de isogenias (secreto) entre dos curvas elípticas (públicas), podría ser roto por un adversario capaz de determinar mediante un proceso eficiente una isogenia entre ambas curvas. Asimismo, la propuesta de firma digital (Patente Microsoft [2]) basa su seguridad en la dificultad de encontrar varias isogenias entre dos curvas elípticas de igual cardinal. Todo ello avala el interés de la investigación en algoritmos que determinen explícitamente isogenias entre dos curvas elípticas.

El grado de una isogenia es el grado de la correspondiente inmersión de los cuerpos de funciones de las curvas elípticas. Además, una isogenia puede obtenerse como composición de isogenias de grado primo. De esta forma, se puede definir un grafo pesado cuyos nodos son curvas elípticas con igual cardinal y cuyas aristas representan isogenias entre curvas con peso el grado de la isogenia. En este contexto, el problema se reduce a encontrar un camino entre dos nodos del grafo.

Galbraith [1] da un algoritmo *de fuerza bruta* para encontrar una tal isogenia. La complejidad de su algoritmo es exponencial y, en general, la isogenia encontrada no es la de grado mínimo posible (problema que el autor considera aún más difícil), sino que produce una de grado *smooth*, es decir, cuyo grado no tiene factores primos mayores que una cierta cota.

En este trabajo presentamos un nuevo algoritmo que produce una isogenia de grado mínimo *smooth* entre dos curvas elípticas dadas. El algoritmo se basa también en un enfoque arbóreo que construye un árbol de isogenias con raíz en E_1, al cual se

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van añadiendo nodos hasta obtener como un nodo del árbol la curva E_2. Entonces, el algoritmo devuelve las isogenias correspondientes a las aristas que se encuentran a lo largo del camino entre E_1 y E_2. Una de las diferencias entre la propuesta de Galbraith [1] y la que presentamos en este trabajo es que las isogenias de nuestro árbol no se construyen de manera aleatoria como en [1] sino que se calculan de tal manera que sus grados son mínimos entre la curva E_1 y la curva imagen por la isogenia que se añade al árbol. Por otra parte, para encontrar un camino, nuestro algoritmo se ciñe a un subgrafo del total, mientras que el algoritmo de Galbraith utiliza el grafo completo.

El algoritmo ha sido implementado en Sage [5] y ejecutado sobre un PC con un procesador Intel Xeon a 3.16 GHz. Hemos limitado el grado máximo de una isogenia a 113, puesto que es el grado primo máximo de los polinomios modulares precalculados con Sage. En la tabla siguiente mostramos algunos resultados. Sobre el cuerpo finito $\mathbb{F}_{1325839561}$ se calcula una isogenia de grado mínimo smooth entre una curva elíptica de j-invariante 1068270565 y una curva elíptica con invariante dado en la primera columna. La segunda columna muestra el grado de tal isogenia mientras que en la tercera se da el tiempo de ejecución del algoritmo.

j-invariante	grado mínimo *smooth*	tiempo (s)
895759403	$2500 = 2^2 \cdot 5^4$	13
1262893225	$5000 = 2^3 \cdot 5^4$	14
609784008	$7500 = 2^2 \cdot 3 \cdot 5^4$	13
89417976	$10000 = 2^4 \cdot 5^4$	14
463359563	$12500 = 2^2 \cdot 5^5$	28
84519525	$15000 = 2^3 \cdot 3 \cdot 5^4$	19
483753258	$17500 = 2^2 \cdot 5^4 \cdot 7$	19
681460722	$20000 = 2^5 \cdot 5^4$	16
99710326	$30000 = 2^4 \cdot 3 \cdot 5^4$	34
29982396	$40000 = 2^6 \cdot 5^4$	37

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Graph Operations and Laplacian Eigenpolytopes

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Abstract. We introduce the *Laplacian eigenpolytopes* ("*L-polytopes*") associated to a simple undirected graph G, investigate how they change under basic operations such as taking the union, join, complement, line graph and cartesian product of graphs, and show how several "famous" polytopes arise as L-polytopes of "famous" graphs.

Eigenpolytopes have been previously introduced by Godsil, who studied them in detail in the context of distance-regular graphs. Our focus on the Laplacian matrix, as opposed to the adjacency matrix of G, permits simpler proofs and descriptions of the result of operations on not necessarily distance-regular graphs. Additionally, it motivates the study of new operations on polytopes, such as the Kronecker product.

Thus, we open the door to a detailed study of how combinatorial properties of G are reflected in its L-polytopes. Subsequent papers will use these tools to construct interesting polytopes from interesting graphs, and vice versa.

Key words: spectrum; Laplacian; polytope; eigenvalue

Let $G = (V, E)$ be a simple graph with $n = |V|$ nodes. Its *adjacency matrix* A is the $n \times n$ matrix with $A_{ij} = 1$ when nodes i and j are adjacent and $A_{ij} = 0$ otherwise. Its *degree matrix* is the diagonal matrix D that collects the degrees of the nodes: $D_{ii} = \deg(i)$. The (ordinary) *spectrum* $\mathrm{Spec}(G)$ of G is the multiset of eigenvalues of A, and the *Laplacian spectrum* $\mathrm{LSpec}(G)$ of G is the multiset of eigenvalues of the *Laplacian Matrix* $L = D - A$ of G. We call these latter eigenvalues the *Laplacian eigenvalues* of G, or *L-values*[1] for short.

By definition, λ is an L-value of G if and only if there exists a non-zero vector $x \in \mathbb{R}^n$ such that $Lx = \lambda x$. In such a situation, we say that x is an *L-vector* of G corresponding to λ, or that x is a *λ-L-vector*. The vector subspace $\mathrm{LE}_\lambda = \ker(L - \lambda\mathbf{I}) \subset \mathbb{R}^n$ of all L-vectors corresponding to λ is the *Laplacian eigenspace* or *L-space* of λ. Its dimension is the multiplicity of the L-value λ. The ordinary eigenspaces of G arise by replacing L by A and LE by $\mathcal{E}$.

In 1978, Godsil [2] associated a so-called *eigenpolytope* to each ordinary eigenspace $\mathcal{E}_\mu$ of a graph, and investigated the eigenpolytopes of distance-regular graphs [1,3]. In the

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[1] No relation (yet) with their cousins from number theory.

present paper, we set out to study some of the relationships between the properties of arbitrary graphs G and their corresponding L-polytopes, which are egenpolytopes of L-spaces. In particular, we investigate how operations on G are reflected in the L-polytopes.

Let $\lambda \in \mathrm{LSpec}(G)$ be an L-value of an undirected graph G, let $\mathrm{LE}_\lambda(G)$ be the corresponding L-space, and put $\mathrm{LE}^0_\lambda(G) := \mathrm{LE}_\lambda(G) \cap \mathbf{1}^\perp$.

Definition 1. *Pick a basis $u_1, \ldots, u_m$ of $\mathrm{LE}^0_\lambda(G)$, assemble these column vectors into the $n \times m$ matrix $Z_\lambda(G)$, and call its rows $g_1, \ldots, g_n$, one for each node of G. The* L-polytope *of G belonging to λ is the convex hull $Q_\lambda(G) = \mathrm{conv}\{g_1, \ldots, g_n\} \in \mathbb{R}^m$ of the rows of $Z_\lambda(G)$.*

Of course, the definition of $Q_\lambda(G)$ depends on the choice of basis. However, a change of basis in $\mathrm{LE}^0_\lambda(G)$ yields $\tilde{Z}_\lambda = Z_\lambda M$ and $\tilde{g}_i = M^\top g_i$ for some $M \in \mathrm{GL}_m(\mathbb{R})$, so that $Q_\lambda(G)$ only suffers a linear transformation.

L-polytopes are interesting because of the strong relationship to the combinatorics of the graph that defines them. Thus, tools from convex polytopes can be used to study graphs and viceversa.

Eventually, we would like to obtain a dictionary that translates between graphs and polytopes, and then use this knowledge to describe polytopes with extremal characteristics parting from extremal graphs. As a first step in this direction, the present paper constructs examples such as simplices, cubes, crosspolytopes, the platonic solids and CUT polytopes as L-polytopes, and provides even more motivation (if necessary) to study exciting new constructions like the Kronecker product of polytopes.

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On the tree-depth of random graphs

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Abstract. The tree-depth $\mathrm{td}(G)$ of a graph G is a measure introduced by *Nešetřil* and Ossona de Mendez [5] in the context of bounded expansion classes. The notion of the tree-depth is closely connected to the tree-width. The tree-width of a graph tells us how similar is G to a tree, while the tree-depth takes also into account the height of the tree.

Bounded expansion classes are defined in terms of shallow minors and its connection to tree–depth is highlighted by the following deep result. The k-th chromatic number of a graph $\chi_k(G)$ is defined as the minimum number of colors needed to color a graph in such a way that the subgraphs H induced by any i classes of colors, $i \leq k$, satisfy $\mathrm{td}(H) \leq i$. Thus $\chi_1(G)$ is the ordinary chromatic number and $\chi_2(G)$ is the so–called star chromatic number. The main theorem in this context states that a class of graphs $\mathcal{C}$ has bounded expansion if and only if $\limsup_{G\in\mathcal{C}} \chi_k(G) < \infty$ for any fixed $k > 0$. This is a clear motivation to study tree-depth.

This parameter has been introduced under numerous names in the literature. It is equivalent to rank function [6], vertex ranking number (or ordered coloring) [2] and upper chromatic number [5].

The following inequalities relate the tree–width and tree–depth of a graph:

$$\mathrm{tw}(G) \leq \mathrm{td}(G) \leq \mathrm{tw}(G)(\log_2 n + 1) \tag{1}$$

Note that there are graphs that have bounded tree-width but unbounded tree-depth, for example trees.

To understand this new parameter, it is useful to know about its behaviour in certain classes of graphs. The main goal of this paper is to analyze how does it behave on random graphs.

We consider the ErdHos-Rényi model $G(n,p)$ for random graph. A *random graph* $G \in G(n,p)$ has n vertices and every pair of vertices is chosen independently to be an edge with probability p.

If $\mathcal{P}$ is a property that a graph can have, we will say that this property holds *asymptotically almost sure (a.s.s.)* for random graphs $G \in G(n,p)$, if

$$\lim_{n\to\infty} \Pr(G \text{ has } \mathcal{P}) = 1$$

We will occasionally make use of the $G(n,m)$ model of random graphs, where a graph with n vertices and m edges is chosen with the uniform distribution. As it is

well–known the two models are closely connected and a.a.s. statements are usually transferred from one model to the other one.

Our first result states the value of tree-depth for dense random graphs.

Theorem 1. *Let $G \in G(n,p)$ be a random graph with $p = \frac{1}{o(n)}$, then G satisfy a.s.s.*

$$\mathrm{td}(G) = n - o(n)$$

This theorem says that when G has super-linear number of edges its tree-depth attains a really large value. Actually our proof of Theorem 1 provides the same result for tree-width. To our knowledge, the tree-width of a dense random graph has not been studied until now.

But, what happens if the number of edges is linear? This case, the sparse case, is solved by the following theorem,

Theorem 2. *Let $G \in G(n,p)$ be a random graph with $p = \frac{c}{n}$, with $c > 0$,*

(1) *if $c < 1$, then a.s.s.* $\mathrm{td}(G) = \Theta(\log \log n)$

(2) *if $c = 1$, then a.s.s.* $\mathrm{td}(G) = \Theta(\log n)$

(3) *if $c > 1$, then a.s.s.* $\mathrm{td}(G) = \Theta(n)$

This last theorem is closely related with a conjecture of Kloks announced in [3] on the linear behaviour of tree-width for random graphs with $c > 1$. This conjecture has been recently proved by Lee, Lee and Oum [4]. Here we give a proof of Theorem 2.(3) which, in view of inequality (1), also provides a simpler proof of Kloks conjecture. Our proof uses, as the one in [4], the same essential result of Benjamini, Kozma and Wormald [1] on the existence of an expander of linear size in a sparse random graph for $c > 1$.

Key words: tree-depth, tree-width, random graphs, threshold function

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On polytopality of Cartesian products of graphs

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Summary

This article focusses on the question of polytopality of certain graphs. In general, the problem is to determine whether a graph can be realized as the graph of a polytope, and in what dimension. This question is well understood until dimension 3: the famous Steinitz' Theorem characterizes graphs of 3-dimensional polytopes. No such characterization can be expected in higher dimension, where the situation is much more complicated. As an illustration, the existence of neighborly polytopes proves that all possible edges can be present in the graph of a 4-polytope. In the first section of our paper, we review some classical results on polytopality of graphs: we first recall the known necessary conditions for a graph to be polytopal and then discuss the case of simple polytopes.

The contribution of the paper to this question concerns a special class of graphs, namely those obtained as Cartesian products of other graphs. The Cartesian product of graphs is defined to be coherent with the Cartesian product of polytopes: the graph of a product of polytopes is the product of their graphs. In particular, the product of two polytopal graphs is automatically polytopal. Two questions then naturally arise:

1. *Dimensional ambiguity of products*: What is the minimal dimension of a realizing polytope of a product of graphs?
2. *Polytopality of non-polytopal graphs*: Are the two factors of a polytopal product of graphs necessarily polytopal?

The first question received much attention in recent literature with the construction of cubical neighborly polytopes, prodsimplicial neighborly polytopes, and the techniques of deformed products of polytopes. In this paper, we provide partial answers to the second question.

Our first result concerns the case of products of regular graphs. These products are regular and it is natural to wonder in which condition they can be graphs of simple polytopes. The answer is given by the following theorem:

Theorem 1. *A product of graphs is the graph of a simple polytope if and only if its factors are.*

In a second stage, we provide a general construction to obtain polytopal products starting from a polytopal graph G and a non-polytopal one H:

Theorem 2. *If G is the graph of a d-dimensional polytope P, and H is the graph of a regular subdivision of an e-dimensional polytope Q, then $G \times H$ is the graph of a $(d+e)$-dimensional polytope.*

As an application of this construction, we obtain for example a 4-dimensional polytope whose graph is the product of two dominos (a domino is the product of a segment by a path). It thus provides an example of a polytopal product of two non-polytopal graphs.

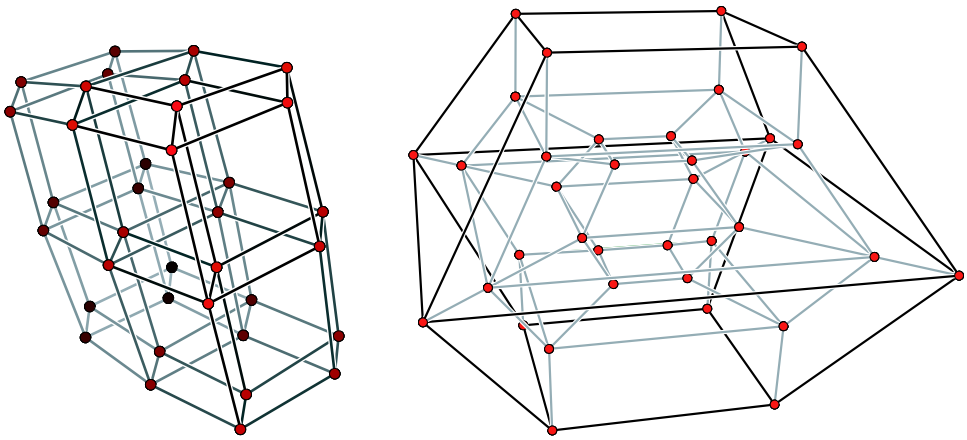

Fig. 1. The graph of the product of two 2-dominos (left) and the Schlegel diagram of a realizing 4-polytope (right).

A generalization of Loday's associahedron

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Summary

This paper focusses on the polytopality of flip graphs on certain geometric structures. For example, the graph whose vertices are all the triangulations of a convex n-gon and whose edges are the flips between them is the graph of the $(n-3)$-dimensional associahedron. This polytope appears under various motivations ranging from geometric combinatorics to algebra, and several different constructions have been proposed.

Flip graphs on two other geometric structures are studied in this paper: the flip graph on pseudotriangulations of a point set in general position in the plane, and the flip graph on multitriangulations of a convex polygon. These two generalizations of the graph of the associahedron have recently been unified by Pilaud and Pocchiola: triangulations, pseudotriangulations and multitriangulations can all be interpreted by duality as certain pseudoline arrangements with contact points supported by a given network.

In this paper, we extend Loday's construction of the associahedron to the general setting of flip graphs on pseudoline arrangements supported by the same network. It results to the construction of a polytope with nice combinatorial properties which realizes the graph of flips restricted to certain pseudoline arrangements. We provide a description of both the vertices and the normal vectors of this polytope.

The polytope associated to a sorting network

Consider a set of n horizontal lines (called *levels*), and place m vertical segments (called *commutators*) joining two consecutive horizontal lines, such that no two commutators have a common endpoint. We fix such a configuration $\mathcal{N}$ that we call *network*. A *pseudoline* is an abscissa monotone path on $\mathcal{N}$ which starts at a level p and ends at the level $n-p$. A *contact point* between two pseudolines is a commutator whose endpoints are contained one in each pseudoline, and a *crossing point* between two pseudolines is a commutator traversed by both pseudolines. A *pseudoline arrangement* is a set of n pseudolines supported by $\mathcal{N}$ such that any two of them have precisely one crossing point, some contact points, and no other intersection. Observe that a pseudoline arrangement supported by $\mathcal{N}$ is completely determined by its $m - \binom{n}{2}$ contact points. We say that a network is *sorting* when it supports some pseudoline arrangements.

There is a natural flip operation which transforms a pseudoline arrangement supported by $\mathcal{N}$ into another one by exchanging the position of a contact point. More precisely, if V is the set of contact points of a pseudoline arrangement Λ supported by $\mathcal{N}$, and if $v \in V$ is a contact point between two pseudolines of Λ which cross at w, then $V' := V \triangle \{v, w\}$ is the set of contact points of another pseudoline arrangement Λ' supported by $\mathcal{N}$.

We associate a polytope $\Omega(\mathcal{N})$ to any sorting network $\mathcal{N}$ as follows:

Definition 1. *Let $\mathcal{N}$ be a sorting network with n levels. The* brick vector *of a pseudoline arrangement Λ supported by $\mathcal{N}$ is the vector $\omega(\Lambda) \in \mathbb{R}^n$ whose ith coordinate is the number of bricks located above the pseudoline of Λ which starts at level i and finishes at level $p - i$. The* brick polytope *$\Omega(\mathcal{N}) \subset \mathbb{R}^n$ of the sorting network $\mathcal{N}$ is the convex hull of the brick vectors of all pseudoline arrangements supported by $\mathcal{N}$.*

Vertex characterization and facet description

Let $\mathcal{N}$ be a sorting network supporting a pseudoline arrangement Λ.

Definition 2. *The* contact graph *of Λ is the multigraph $\Lambda^\#$ with a node for each pseudoline of Λ and an arc for each contact point of Λ oriented from the pseudoline passing above the contact point to the pseudoline passing below it.*

We exploit this contact graph to describe the cone of $\Omega(\mathcal{N})$ at $\omega(\Lambda)$:

Theorem 1. *The cone of the brick polytope $\Omega(\mathcal{N})$ at the brick vector $\omega(\Lambda)$ is precisely the incidence cone $C(\Lambda^\#) := \operatorname{cone}\left\{e_i - e_j \mid (i,j) \in \Lambda^\#\right\}$ of the contact graph of Λ.*

As corollaries of this theorem, we obtain a vertex characterization and a facet description of the brick polytope:

Corollary 1. *The brick vector $\omega(\Lambda)$ is a vertex of the brick polytope $\Omega(\mathcal{N})$ if and only if the contact graph of Λ is acyclic.*

Corollary 2. *The normal vectors of the brick polytope $\Omega(\mathcal{N})$ are precisely all normal vectors of the incidence cones of the contact graphs of the pseudoline arrangements supported by $\mathcal{N}$, that is, the characteristic vectors of the sinks of the directed cuts of these contact graphs.*

Perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes applied to steganography *

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Abstract. Steganography aims to hide secret data imperceptibly into a cover object. We describe a novel method based on $\mathbb{Z}_2\mathbb{Z}_4$-linear codes in which data is embedded by distorting each cover symbol by one unit at most ($\pm$1-steganography). This method solves the problem encountered by the most efficient methods known today, concerning the treatment of boundary values. Moreover, a generalization of the product perfect codes technique is applied to this method, and its performance is evaluated.

Key words: Additive codes, data hiding, perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, steganography.

1 Introduction

Steganography hides data imperceptibly within innocuous cover media. The secret message $\mathbf{s} \in \{1, \ldots, M\}$ produces a composite sequence $\mathbf{y} = f(\mathbf{x}, \mathbf{s})$, obtained by distorting the cover object $\mathbf{x} = (x_1, \ldots, x_N)$, where $x_i, y_i \in \aleph = \{0, 1, \ldots, 2^B - 1\}$ and $B = 8$ for grayscale images. We call *CI-rate* [2] the tuple (D, E), where D is the *average distortion* and E is the *embedding rate*. We focus on "$\pm$1-steganography", that is, $y_i = x_i + c$, where $c \in \{0, +1, -1\}$. Previous papers on this issue involved ternary codes [4,2] but dealt inefficiently with boundary grayscale values (0 and 2^B-1) because changes of magnitude 2 (+2, −2) were applied.

Any non-empty subgroup $\mathcal{C}$ of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is a $\mathbb{Z}_2\mathbb{Z}_4$-*additive code* [1]. Let $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \to \mathbb{Z}_2^n$ be the *extended Gray map* given by applying the usual *Gray map* $\phi(0) = (0,0)$, $\phi(1) = (0,1)$, $\phi(2) = (1,1)$, and $\phi(3) = (1,0)$ to the quaternary coordinates. The binary image $C = \Phi(\mathcal{C})$ of a $\mathbb{Z}_2\mathbb{Z}_4$-*additive code* $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-*linear code* and may not be linear. Code $\mathcal{C}$ is said to be *perfect* if $C = \Phi(\mathcal{C})$ is a perfect binary code.

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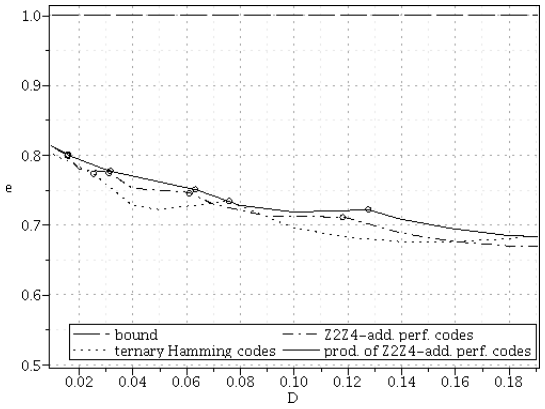
2 Main results

Theorem 1 *For $m \geq 4$, the CI-rate given by the method based on $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes improves the CI-rate obtained by direct sum of ternary Hamming codes with the same average distortion.*

The magnitude of embedding changes of this new method is under no circumstances greater than one.

There is an efficient steganographic method [3] based on the Kronecker product of the generator matrices of two Hamming codes. In the following figure we can see how, for a given distortion D, if we take the product of more than two perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, its normalized embedding rate e improves perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, and the latter improves in turn ternary Hamming codes.



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El problema de Frobenius para algunos semigrupos numéricos de dimensión de inmersión tres ⋆

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Resumen. Sea S un semigrupo numérico de dimensión de inmersión tres tal que sus generadores minimales son números primos relativos dos a dos. Entonces existen números enteros positivos a, b, c, d tales que $\mathrm{mcd}\{a, b\} = \mathrm{mcd}\{a, c\} = \mathrm{mcd}\{b, d\} = 1$, $c \in \{2, \ldots, a-1\}$ y $cb - da > 0$, de forma que $S = \langle a, b, cb - da\rangle$. Si además se supone que el intervalo $\left[\frac{a}{c}, \frac{b}{d}\right]$ contiene algún número entero, se obtienen fórmulas para el género, el número de Frobenius y el conjunto de pseudo-números de Frobenius de S en función de a, b, c, d.

Palabras clave: Problema de Frobenius, semigrupos numéricos, conjunto de Apéry.

1 Resumen ampliado

Un *semigrupo númerico* es un conjunto $S \subseteq \mathbb{N}$ tal que es cerrado para la suma, $0 \in S$ y $\mathbb{N} \setminus S$ es finito. Los elementos de $\mathbb{N} \setminus S$ son los *huecos* de S y al cardinal de dicho conjunto se le denomina *género* de S, denotándose por $\mathrm{g}(S)$. El *número de Frobenius* de S es el mayor entero que no pertenece a S y se denota por $\mathrm{F}(S)$.

Es bien conocido (véase [5]) que para cada semigrupo numérico S existe un único conjunto finito $G \subseteq S$ que genera a S y ningún subconjunto propio de G genera a S. Se dice que G es el *sistema minimal de generadores* de S.

El problema de Frobenius (véase [3]) consiste en encontrar fórmulas que permitan calcular, en términos del sistema minimal de generadores de un semigrupo numérico dado, el número de Frobenius y el género de dicho semigrupo numérico. Este problema fue resuelto por Sylvester y Curran Sharp (véase [6]) en el caso de dimensión de inmersión dos. Por contra, para el caso de dimensión tres, Curtis demostró en [2] que es imposible dar una expresión polinomial que permita calcular el número de Frobenius. La motivación de este trabajo es proporcionar una familia amplia de semigrupos numéricos de dimensión de inmersión tres para la que se resuelve tal problema y, además, la solución obtenida tiene una expresión sencilla.

Para llevar a cabo este trabajo nos centraremos en el caso de semigrupos numéricos cuyos generadores minimales n_1, n_2, n_3 son números primos relativos dos a dos. La

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herramienta principal será el *conjunto de Apéry* de m en S (véase [1]). Otro concepto a utilizar será el de *pseudo-número de Frobenius* (véase [4]), junto con el de conjunto de pseudo-números de Frobenius de S (que se denota por $\mathrm{PF}(S)$).

Considerando la familia de semigrupos

$$\mathcal{F} = \{\langle a, b, cb - da\rangle \mid a, b, c, d \in \mathbb{N} \setminus \{0\},\ \mathrm{mcd}\{a, b\} = \mathrm{mcd}\{a, c\} =$$

$$\mathrm{mcd}\{b, d\} = 1,\ 2 \leq c \leq a - 1,\ cb - da > 0\}\,,$$

el principal resultado de este trabajo es el siguiente teorema.

Teorema 1. *Sea el semigrupo numérico* $S = \langle a, b, cb - da\rangle \in \mathcal{F}$. *Si se verifica la desigualdad* $cb - da \geq d((-a) \bmod c)$ *entonces*

$$\mathrm{Ap}(S, a) = \left\{\alpha b - \left\lfloor \frac{\alpha}{c} \right\rfloor da \mid \alpha \in \{0, \ldots, a - 1\}\right\}.$$

Como consecuencia de tal resultado, obtenemos $\mathrm{F}(S)$, $\mathrm{g}(S)$ y $\mathrm{PF}(S)$.

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(La lista de referencias está ampliada en el trabajo presentado.)